Eur. Phys. J. C 21, 659–678 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100755

THE EUROPEAN PHYSICAL JOURNAL C

Resonance estimates of $\mathcal{O}(p^6)$ low-energy constants and QCD short-distance constraints

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Received: 4 June 2001 /
Published online: 31 August 2001 –

Abstract. Starting from the study of the low-energy and high-energy behaviors of the QCD three-point functions $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$, several $\mathcal{O}(p^6)$ low-energy constants of the chiral Lagrangian are evaluated within the framework of the lowest meson dominance (LMD) approximation to the large-N_C limit of QCD. In certain cases, values that differ substantially from estimates based on a resonance Lagrangian are obtained. It is pointed out that the differences arise through the fact that QCD short-distance constraints are in general not correctly taken into account in the approaches using resonance Lagrangians. We discuss the implications of our results for the $\mathcal{O}(p^6)$ counterterm contributions to the vector form factor of the pion and to the decay $\pi \to e\nu_e\gamma$, and for the pion–photon–photon transition form factor.

1 Introduction

In the chiral limit, the lightest pseudoscalar states of the hadronic spectrum become the octet of massless Goldstone bosons resulting from the spontaneous breaking of the chiral $SU(3)_{\mathrm{L}} \times SU(3)_{\mathrm{R}}$ global symmetry of the QCD Lagrangian towards its diagonal subgroup $SU(3)_V$ of vector symmetries. This well-known fact [1,2] allows one to describe the interactions of the light pseudoscalar mesons at low energies in terms of an effective Lagrangian [3,4]. The latter involves the pseudoscalar fields, described by a unitary matrix $U(x)$, and transforming under a nonlinear representation of the chiral symmetry group, as well as the sources, $v_\mu(x)$, $a_\mu(x)$, of the light-quark vector and axial currents, and $s(x)$, $p(x)$, of the scalar and pseudoscalar densities of QCD [4]. Matrix elements of these currents between pseudoscalar states, or scattering amplitudes involving these light states only, can be computed in a systematic way in the low-energy theory as long as all momentum transfers p^2 are sufficiently small, $p^2 \ll \Lambda_H^2$, where $\Lambda_H \sim 1\,\mathrm{GeV}$ is the typical scale at which the non-Goldstone bound states of QCD are formed. Since the (running) light-quark masses $m_q(\mu)$ are also small as compared to this scale, $m_{u,d,s}(\Lambda_H) \ll \Lambda_H$, the effective Lagrangian $\mathcal{L}^{\mathrm{eff}}$ can be organized as an expansion in powers of derivatives of the field $U(x)$ and powers of the light-quark masses,

$$\mathcal{L}^{\mathrm{eff}} = \sum_{k,l} \mathcal{L}_{(k,l)},$$

with

$$\mathcal{L}_{(k,l)} \sim \left(\frac{p}{\Lambda_H}\right)^k \left(\frac{m_q}{\Lambda_H}\right)^l . \tag{1.1}$$

The presently available studies on the structure of the low-energy effective Lagrangian involve, beyond the lowest order terms, the pieces $\mathcal{L}_{(4,0)}$, $\mathcal{L}_{(2,1)}$ and $\mathcal{L}_{(0,2)}$ [4], $\mathcal{L}_{(6,0)}$ and $\mathcal{L}_{(4,1)}$ [5,6], $\mathcal{L}_{(2,2)}$ and $\mathcal{L}_{(0,3)}$ [7,5,6], as well as $\mathcal{L}_{(0,4)}$ [8]. The structure of each $\mathcal{L}_{(k,l)}$ is entirely fixed by the chiral symmetry properties of QCD [9], but involves coefficients, the so called low-energy constants[1], which are not determined from symmetry requirements alone. The predictive power of the effective theory therefore hinges to quite some extent on the knowledge of these low-energy constants. At $\mathcal{O}(p^4)$, the values of most of the low-energy constants were extracted from data. The proliferation of low-energy constants at $\mathcal{O}(p^6)$ makes such an approach unrealistic.

On the other hand, it is a general property that these low-energy constants correspond to the coefficients of the Taylor expansion, with respect to the momenta, of some QCD correlation functions, once the singularities (poles and discontinuities) associated with the contributions of low-momentum pseudoscalar intermediate states have been subtracted. The characteristic feature of the Green's functions that are actually involved is that they are *order parameters* of the spontaneous breaking of chiral sym-

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[1] Some of the counterterms involving the external sources only and no pseudoscalar fields actually rather correspond to "high-energy constants", since they describe the (perturbative) short-distance ambiguities of some QCD correlators. At $\mathcal{O}(p^4)$, this concerns the constants H_1 and H_2 of [4]. We exclude this type of counterterms from the present discussion

metry. Thus, they do not receive contributions from perturbative QCD at large momentum transfers, but rather exhibit a smooth behavior at short distances. The low-energy constants are thus expected to be sensitive to the physics in the intermediate energy region; that is, to the spectrum of mesonic resonances in the mass region around the hadronic scale Λ_H. This basic observation underlies, in some way or another, most attempts to estimate the values of the low-energy constants from resonance data (for an introduction to the vast bibliography on this subject, we refer the reader to the review articles [10–12]).

It has become customary to describe the effects of resonance states within a Lagrangian framework, by introducing, besides the Goldstone boson field $U(x)$, additional local fields associated with the meson states. While there exists a systematic way [13] to construct fields which have the appropriate transformation properties under chiral transformations and invariant Lagrangians, there is however no restriction from chiral symmetry as to the number of fields and the order of derivatives thereof involved in the terms which describe the interaction among resonances or between the resonances and the Goldstone bosons. In addition, the construction of [13] leaves open the question of the choice of the Lorentz group representation for the field describing a meson state of a given spin. This lack of restrictions has led to many proposals concerning a Lagrangian description of interacting Goldstone bosons and mesons (the literature on this subject can be traced back from several reviews [14–16] and from the articles of [17–19]). It has been pointed out in [20] that restrictions actually can be introduced, via the QCD *short-distance* properties of the relevant Green's functions or form factors, and by requiring that these properties are satisfied by the same objects constructed with the help of the Lagrangian involving resonances. This aspect has been taken into account in a rather systematic manner at the level of the $\mathcal{O}(p^4)$ counterterms [20]. Although resonance estimates have been given for several $\mathcal{O}(p^6)$ low-energy constants, the importance of implementing the appropriate QCD short-distance constraints has not always been stressed. The purpose of the present study is precisely to address this aspect.

Although quite practical and useful, since it guarantees that some general properties (locality, analyticity) of quantum field theories are correctly taken into account, a Lagrangian formalism is not absolutely necessary in order to estimate the contributions of the QCD resonance states to the low-energy constants. In this article, we shall in fact consider a different approach, working directly with the appropriate Green's functions. We shall however retain the general features and assumptions that underlie the Lagrangian approach, and that we briefly recall. First, one usually considers zero-width resonances, and takes into account the contributions from one-resonance states, which produce only poles in the corresponding correlation functions. There exists a well-defined framework where this kind of restrictions arises in a natural way [21], namely the large-N_C limit of QCD [22]. On the other hand, working in this limit still requires one to consider, in each channel, an infinite number of resonances, with masses and coupling constants adjusted such as to reproduce the QCD perturbative continuum at high momentum transfers. Of course, such a point of view is rather ambitious, since it amounts to solving QCD in the large-N_C limit, a reputedly difficult task. We shall adopt a more modest attitude, assuming that in each channel a few lowest-lying resonances give already the main contribution (approaches involving an infinite number of zero-width resonances, with various additional assumptions about their spacing, decay constants, etc., can be found in [23–25]). The number of resonances to be considered in each channel will be taken as the minimal (finite) number necessary to satisfy the requirements set by, say, the leading QCD short-distance constraints for the Green's functions under consideration, and possibly other constraints that one may wish to impose. This *minimal hadronic ansatz* (MHA) approximation may be well justified in the case of Green's functions which are order parameters, free of perturbative contributions (for Green's functions which are not order parameters, the QCD continuum contribution has to be included as well; see [26, 27]). In fact, in many cases, this minimal hadronic ansatz can be reduced to retaining, in each channel, a single resonance. At the $\mathcal{O}(p^4)$ level, this *lowest meson dominance* (LMD) approximation to large-N_C QCD has been tested successfully in several instances [17,18,20,26]. As we shall see in the examples treated in the present work, depending on the constraints one wishes to implement on the Green's functions under consideration, this simplest LMD approximation may however not always be sufficient. A second feature common to the Lagrangian and to the LMD or MHA approximations is the fact that the estimates of the low-energy constants do not reproduce their scale dependence. The latter, which comes from Goldstone boson loop contributions to the relevant Green's functions, is a next-to-leading order effect in the $1/N_C$ expansion, and lies thus beyond the approximation considered. We shall adopt the usual point of view that the estimates furnished by this type of approach corresponds to the values of the low-energy constants at the typical scale, say Λ_H, set by these resonance states [18].

Approaches which do not rely on a resonance Lagrangian have been used before at the $\mathcal{O}(p^6)$ level for two-point functions [28–30]. In these studies, the relevant low-energy constants have often been expressed through (superconvergent) dispersion integrals of the corresponding spectral densities, which were then evaluated using available data. It seems difficult to follow similar lines in the case of three-point or higher Green's functions. Not only are their analyticity properties far more complicated, but the corresponding spectral densities are in general not experimental observables. Studies of three-point functions similar to the lines we follow here can be found in [31–33], although the discussion of their short-distance properties is less complete than the one presented below.

In this article, we shall concentrate on a certain subset of $\mathcal{O}(p^6)$ low-energy constants contributing to $\mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(6,0)}$, and corresponding to the three-point functions $\langle VAP \rangle$, $\langle VVP \rangle$ and $\langle AAP \rangle$ (see the beginning of Sect. 2

for the precise definitions). There are several reasons for this specific choice. First, these correlators have been considered before in the literature, so that quite some information concerning them is already available. Second, although the approach that we shall follow here is, in principle, applicable to other correlation functions as well, these Green's functions have a certain degree of simplicity, which makes them particularly valuable for illustrating our point of view. Finally, these Greens functions also play a role in the evaluation of some of the counterterms that arise in the calculation of electromagnetic contributions to the pseudoscalar masses [34] or of radiative corrections to semileptonic decays of the pseudoscalar mesons [35,36] within an effective Lagrangian framework [37].

The remaining material of the present article is organized as follows. In Sect. 2, we define the relevant QCD Green's function and study their long-distance properties in the chiral limit and at leading order in the $1/N_C$ expansion. In particular, we identify the low-energy constants related to these correlators. Section 3 is devoted to an extensive discussion of the leading short-distance properties of these Green's functions within the same framework. In Sect. 4, we construct some simple ansätze, in terms of a finite number of narrow resonances, which correctly reproduce the short-distance constraints. These are used to determine the corresponding low-energy constants in Sect. 5. We then compare our results with those obtained from a Lagrangian involving resonances [38] which has often been employed in the literature to estimate the low-energy constants at order p^6 (Sect. 6), and point out that this resonance Lagrangian does not correctly incorporate the necessary short-distance properties (Sect. 7). In Sect. 8, we present several applications. Conclusions and additional discussions can be found in Sect. 9. Appendix A contains some technical details relevant for the discussion in Sect. 4 and Appendix B gives the expression of the resonance Lagrangian of [38].

2 Long-distance properties from chiral symmetry

We consider, in the three flavor chiral limit, the momentum space QCD three-point functions

$$\begin{aligned}(\Pi_{VAP})^{abc}_{\mu\nu}(p,q) &= \int \mathrm{d}^4x \int \mathrm{d}^4y \mathrm{e}^{\mathrm{i}(p\cdot x+q\cdot y)} \\ &\quad \times \langle 0|T\{V^a_\mu(x)A^b_\nu(y)P^c(0)\}|0\rangle, \\ (\Pi_{VVP})^{abc}_{\mu\nu}(p,q) &= \int \mathrm{d}^4x \int \mathrm{d}^4y \mathrm{e}^{\mathrm{i}(p\cdot x+q\cdot y)} \\ &\quad \times \langle 0|T\{V^a_\mu(x)V^b_\nu(y)P^c(0)\}|0\rangle, \\ (\Pi_{AAP})^{abc}_{\mu\nu}(p,q) &= \int \mathrm{d}^4x \int \mathrm{d}^4y \mathrm{e}^{\mathrm{i}(p\cdot x+q\cdot y)} \\ &\quad \times \langle 0|T\{A^a_\mu(x)A^b_\nu(y)P^c(0)\}|0\rangle, \end{aligned} \tag{2.1}$$

involving the octet vector and axial currents,

$$V^a_\mu(x) = \left(\bar\psi\gamma_\mu\frac{\lambda^a}{2}\psi\right)(x),$$

$$A^a_\mu(x) = \left(\bar\psi\gamma_\mu\gamma_5\frac{\lambda^a}{2}\psi\right)(x),$$

as well as the octet pseudoscalar density

$$P^a(x) = \left(\bar\psi \mathrm{i}\gamma_5\frac{\lambda^a}{2}\psi\right)(x).$$

These three-point functions satisfy the following chiral Ward identities,

$$\begin{aligned} p^\mu(\Pi_{VAP})^{abc}_{\mu\nu}(p,q) &= \langle\overline{\psi}\psi\rangle_0 f^{abc}\left[\frac{q_\nu}{q^2} - \frac{(p+q)_\nu}{(p+q)^2}\right], \\ q^\nu(\Pi_{VAP})^{abc}_{\mu\nu}(p,q) &= \langle\overline{\psi}\psi\rangle_0 f^{abc}\frac{(p+q)_\mu}{(p+q)^2}, \\ p^\mu(\Pi_{VVP})^{abc}_{\mu\nu}(p,q) &= 0, q^\nu(\Pi_{VVP})^{abc}_{\mu\nu}(p,q) = 0, \\ p^\mu(\Pi_{AAP})^{abc}_{\mu\nu}(p,q) &= 0, q^\nu(\Pi_{AAP})^{abc}_{\mu\nu}(p,q) = 0, \end{aligned} \tag{2.2}$$

where $\langle\overline{\psi}\psi\rangle_0$ denotes the single flavor bilinear quark condensate in the chiral limit. The general solution of these Ward identities, taking into account the invariances of QCD under $SU(3)_V$, parity and time reversal transformations (the latter being responsible for the absence of structures of the type d^{abc} in the case of Π_{VAP}, or of structures of the type f^{abc} in the cases of Π_{VVP} and Π_{AAP}), read

$$\begin{aligned} &(\Pi_{VAP})^{abc}_{\mu\nu}(p,q) \\ &\quad = f^{abc}\Bigg\{\langle\overline{\psi}\psi\rangle_0\left[\frac{(p+2q)_\mu q_\nu}{q^2(p+q)^2} - \frac{\eta_{\mu\nu}}{(p+q)^2}\right] \\ &\quad +P_{\mu\nu}(p,q)\mathcal{F}(p^2,q^2,(p+q)^2) \\ &\quad +Q_{\mu\nu}(p,q)\mathcal{G}(p^2,q^2,(p+q)^2)\Bigg\}, \\ &(\Pi_{VVP})^{abc}_{\mu\nu}(p,q) = \epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta d^{abc}\mathcal{H}_V(p^2,q^2,(p+q)^2), \\ &(\Pi_{AAP})^{abc}_{\mu\nu}(p,q) = \epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta d^{abc}\mathcal{H}_A(p^2,q^2,(p+q)^2) \\ &. \end{aligned} \tag{2.3}$$

Here, $\eta_{\mu\nu}$ denotes the metric tensor in flat Minkowski space with signature $(+,-,-,-)$ and we use the conventions $\epsilon_{0123} = 1$ for the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The transverse tensors $P_{\mu\nu}$ and $Q_{\mu\nu}$ are defined by

$$\begin{aligned} P_{\mu\nu}(p,q) &= q_\mu p_\nu - (p\cdot q)\eta_{\mu\nu}, \\ Q_{\mu\nu}(p,q) &= p^2 q_\mu q_\nu + q^2 p_\mu p_\nu - (p\cdot q)p_\mu q_\nu - p^2q^2\eta_{\mu\nu}. \end{aligned}$$

Due to Bose–Einstein symmetry, the invariant functions $\mathcal{H}_{V,A}$ have the property

$$\mathcal{H}_{V,A}(p^2,q^2,(p+q)^2) = \mathcal{H}_{V,A}(q^2,p^2,(p+q)^2). \tag{2.4}$$

The behavior of these invariant functions at small momentum transfers is constrained by the presence of singularities arising from Goldstone boson intermediate states. Here we are interested in the limit where the number of colors N_C becomes infinite. In this limit, the contributions from one-particle intermediate states dominate, so that at low energies we only need to keep the corresponding Goldstone boson poles and the polynomial terms involving the counterterms. In the even intrinsic parity case, we use the basis of [6] for the $\mathcal{O}(p^6)$ counterterms, and we obtain

$$\begin{aligned}
\mathcal{F}^{\rm ChPT}(p^2,q^2,(p+q)^2) &= \frac{4\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2} \\
&\times\left[L_9+L_{10}+\left(C_{78}-\frac{5}{2}C_{88}-C_{89}+3C_{90}\right)p^2\right. \\
&+\left(C_{78}-2C_{87}+\frac{1}{2}C_{88}\right)q^2 \\
&\left.+\left(C_{78}+4C_{82}-\frac{1}{2}C_{88}\right)(p+q)^2\right]+\cdots, \\
\mathcal{G}^{\rm ChPT}(p^2,q^2,(p+q)^2) &= \frac{4\langle\overline{\psi}\psi\rangle_0}{F_0^2q^2(p+q)^2} \\
&\times[L_9+2(-C_{88}+C_{90})p^2+(2C_{78}-C_{89}+C_{90})q^2 \\
&-2C_{90}(p+q)^2]+\cdots,
\end{aligned} \tag{2.5}$$

where the ellipses stand for higher order contributions. For the two correlators of odd intrinsic parity, we use the counterterm Lagrangian of [5], in terms of which we find

$$\begin{aligned}
\mathcal{H}_V^{\rm ChPT}(p^2,q^2,(p+q)^2) &= -\frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2} \\
&\times\left[-\frac{N_C}{8\pi^2}+(4A_2-16A_3)(p^2+q^2)\right. \\
&\left.+(-4A_2+8A_3+16A_4)(p+q)^2\right]+\cdots, \\
\mathcal{H}_A^{\rm ChPT}(p^2,q^2,(p+q)^2) &= -\frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2} \\
&\times\left[-\frac{N_C}{24\pi^2}+(4A_{11}+4A_{23}-16A_{24})(p^2+q^2)\right. \\
&+(-12A_{11}-16A_{17}-4A_{23}+8A_{24}+16A_{25}) \\
&\left.\times(p+q)^2\right]+\cdots.
\end{aligned} \tag{2.6}$$

We have thus identified the set of low-energy constants that describe the long-distance behavior of the $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$ three-point functions.

3 Short-distance analysis

We next study the properties of the $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$ correlators at short distances. These will be conditioned by the fact that the three Green's functions under consideration are order parameters of chiral symmetry. Therefore, they vanish to all orders in perturbative QCD in the chiral limit, so that their behavior at short distances is smoother than expected from naive power counting arguments. Two limits are of interest. In the first case, the two momenta become simultaneously large, which in position space describes the situation where the space-time arguments of the three operators tend towards the same point at the same rate. Our analysis will be restricted to the leading terms[2], and the expressions below hold up to corrections of order $\mathcal{O}(\alpha_s)$. We obtain

$$\begin{aligned}
&\lim_{\lambda\to\infty}(\Pi_{VAP})^{abc}_{\mu\nu}(\lambda p,\lambda q) = \frac{\langle\overline{\psi}\psi\rangle_0}{\lambda^2}f^{abc}\frac{1}{p^2q^2(p+q)^2} \\
&\times\left\{p^2(p+2q)_\mu q_\nu-\eta_{\mu\nu}p^2q^2\right. \\
&\left.+\frac{1}{2}[p^2-q^2-(p+q)^2]P_{\mu\nu}-Q_{\mu\nu}\right\}+\mathcal{O}\left(\frac{1}{\lambda^4}\right), \\
&\lim_{\lambda\to\infty}(\Pi_{VVP})^{abc}_{\mu\nu}(\lambda p,\lambda q) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2\lambda^2}d^{abc}\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \\
&\times\frac{q^2+p^2+(p+q)^2}{p^2q^2(p+q)^2}+\mathcal{O}\left(\frac{1}{\lambda^4}\right), \\
&\lim_{\lambda\to\infty}(\Pi_{AAP})^{abc}_{\mu\nu}(\lambda p,\lambda q) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2\lambda^2}d^{abc}\epsilon_{\mu\nu\alpha\beta}p^\alpha q^\beta \\
&\times\frac{q^2+p^2-(p+q)^2}{p^2q^2(p+q)^2}+\mathcal{O}\left(\frac{1}{\lambda^4}\right).
\end{aligned} \tag{3.1}$$

One concludes that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\mathcal{F}((\lambda p)^2,(\lambda q)^2,(\lambda p+\lambda q)^2) \\
&\quad=\frac{1}{2\lambda^4}\langle\overline{\psi}\psi\rangle_0\frac{p^2-q^2-(p+q)^2}{p^2q^2(p+q)^2}+\mathcal{O}\left(\frac{1}{\lambda^6}\right),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\lim_{\lambda\to\infty}\mathcal{G}((\lambda p)^2,(\lambda q)^2,(\lambda p+\lambda q)^2) \\
&\quad=-\frac{1}{\lambda^6}\frac{\langle\overline{\psi}\psi\rangle_0}{p^2q^2(p+q)^2}+\mathcal{O}\left(\frac{1}{\lambda^8}\right),
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\lim_{\lambda\to\infty}\mathcal{H}_{V,A}((\lambda p)^2,(\lambda q)^2,(\lambda p+\lambda q)^2) \\
&\quad=-\frac{1}{2\lambda^4}\langle\overline{\psi}\psi\rangle_0\frac{p^2+q^2\pm(p+q)^2}{p^2q^2(p+q)^2}+\mathcal{O}\left(\frac{1}{\lambda^6}\right).
\end{aligned} \tag{3.4}$$

Notice that since the $\langle\overline{\psi}\psi\rangle_0$ condensate and the pseudoscalar density $P^a(x)$ have the same anomalous dimensions, the leading short-distance behavior exhibited in these expressions is canonical, the corresponding Wilson coefficients have no anomalous dimensions.

The second situation of interest corresponds to the case where the relative distance between only two of the three operators involved becomes small. It so happens that the corresponding behaviors in momentum space involve, apart from the correlator $\langle AP\rangle$ which, in the chiral limit, is saturated by the single-pion intermediate state,

$$\int {\rm d}^4x{\rm e}^{{\rm i}p\cdot x}\langle 0|T\{A^a_\mu(x)P^b(0)\}|0\rangle=\delta^{ab}\langle\overline{\psi}\psi\rangle_0\frac{p_\mu}{p^2},$$

[2] In the case of the $\langle VAP\rangle$ correlator, the subleading term in the short-distance expansion, involving the mixed condensate, can be found in [33]

the two-point function $\langle VT\rangle$ of the vector current and the antisymmetric tensor density,

$$\delta^{ab}(\Pi_{VT})_{\mu\rho\sigma}(p) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}p\cdot x}\langle 0|T\left\{V_\mu^a(x)\left(\overline{\psi}\sigma_{\rho\sigma}\frac{\lambda^b}{2}\psi\right)(0)\right\}|0\rangle,$$

with $\sigma_{\rho\sigma} = (\mathrm{i}/2)[\gamma_\rho,\gamma_\sigma]$ (the similar correlator between the axial current and the tensor density vanishes as a consequence of invariance under charge conjugation). Conservation of the vector current and invariance under parity then give

$$(\Pi_{VT})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma} - p_\sigma\eta_{\mu\rho})\Pi_{VT}(p^2).$$

The leading short-distance behavior of this two-point function reads (see also [39])

$$\lim_{\lambda\to\infty}\Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2}\frac{\langle\overline{\psi}\psi\rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right). \quad (3.5)$$

For the $\langle VAP\rangle$ correlator, we then find

$$\lim_{\lambda\to\infty}(\Pi_{VAP})^{abc}_{\mu\nu}(\lambda p, q-\lambda p) \quad (3.6)$$
$$= -\frac{1}{\lambda}f^{abc}\langle\overline{\psi}\psi\rangle_0\frac{p_\mu q_\nu + p_\nu q_\mu - (p\cdot q)\eta_{\mu\nu}}{p^2q^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right),$$

$$\lim_{\lambda\to\infty}(\Pi_{VAP})^{abc}_{\mu\nu}(\lambda p, q) = \frac{1}{\lambda}f^{abc}\langle\overline{\psi}\psi\rangle_0\frac{p_\mu q_\nu}{p^2q^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (3.7)$$

and

$$\lim_{\lambda\to\infty}(\Pi_{VAP})^{abc}_{\mu\nu}(p, \lambda q) \quad (3.8)$$
$$= \frac{1}{\lambda}f^{abc}\frac{p_\nu q_\mu - (p\cdot q)\eta_{\mu\nu}}{q^2}\Pi_{VT}(p^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right).$$

For the $\langle VVP\rangle$ correlator we obtain the results

$$\lim_{\lambda\to\infty}(\Pi_{VVP})^{abc}_{\mu\nu}(\lambda p, q-\lambda p)$$
$$= -\frac{1}{\lambda}d^{abc}\langle\overline{\psi}\psi\rangle_0\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma\frac{1}{p^2q^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (3.9)$$

$$\lim_{\lambda\to\infty}(\Pi_{VVP})^{abc}_{\mu\nu}(\lambda p, q)$$
$$= \frac{1}{\lambda}d^{abc}\epsilon_{\mu\nu\rho\sigma}\frac{p^\rho q^\sigma}{p^2}\Pi_{VT}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (3.10)$$

and for the $\langle AAP\rangle$ correlator,

$$\lim_{\lambda\to\infty}(\Pi_{AAP})^{abc}_{\mu\nu}(\lambda p, q-\lambda p)$$
$$= -\frac{1}{\lambda}d^{abc}\langle\overline{\psi}\psi\rangle_0\epsilon_{\mu\nu\rho\sigma}p^\rho q^\sigma\frac{1}{p^2q^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad (3.11)$$

$$\lim_{\lambda\to\infty}(\Pi_{AAP})^{abc}_{\mu\nu}(\lambda p, q) = \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (3.12)$$

In terms of the invariant functions $\mathcal{F}$ and $\mathcal{G}$, the constraint (3.6) yields

$$\lim_{\lambda\to\infty}\mathcal{F}((\lambda p)^2, (q-\lambda p)^2, q^2)$$
$$= \frac{\langle\overline{\psi}\psi\rangle_0}{\lambda^2p^2}\left[\mathcal{F}^{(0)}(q^2) + \frac{1}{\lambda}\frac{p\cdot q}{p^2}\mathcal{F}^{(1)}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right],$$
$$\lim_{\lambda\to\infty}\mathcal{G}((\lambda p)^2, (q-\lambda p)^2, q^2) \quad (3.13)$$
$$= \frac{\langle\overline{\psi}\psi\rangle_0}{(\lambda^2p^2)^2}\left[\mathcal{G}^{(0)}(q^2) + \frac{1}{\lambda}\frac{p\cdot q}{p^2}\mathcal{G}^{(1)}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right],$$

together with

$$\mathcal{F}^{(0)}(q^2) - \mathcal{G}^{(0)}(q^2) = \frac{1}{q^2},$$
$$\mathcal{F}^{(1)}(q^2) - \mathcal{G}^{(1)}(q^2) + \mathcal{G}^{(0)}(q^2) = \frac{2}{q^2}. \quad (3.14)$$

Finally, the following properties

$$\lim_{\lambda\to\infty}\mathcal{F}((\lambda p)^2, q^2, (q+\lambda p)^2) = \mathcal{O}\left(\frac{1}{\lambda^3}\right),$$
$$\lim_{\lambda\to\infty}\mathcal{G}((\lambda p)^2, q^2, (q+\lambda p)^2) = \mathcal{O}\left(\frac{1}{\lambda^4}\right), \quad (3.15)$$

and

$$\lim_{\lambda\to\infty}\mathcal{F}(p^2, (\lambda q)^2, (p+\lambda q)^2)$$
$$= \frac{1}{\lambda^2}\frac{1}{q^2}\Pi_{VT}(p^2) + \mathcal{O}\left(\frac{1}{\lambda^3}\right),$$
$$\lim_{\lambda\to\infty}\mathcal{G}(p^2, (\lambda q)^2, (p+\lambda q)^2) = \mathcal{O}\left(\frac{1}{\lambda^4}\right), \quad (3.16)$$

must also be satisfied. For the invariant functions $\mathcal{H}_V$ and $\mathcal{H}_A$ we obtain the constraints

$$\lim_{\lambda\to\infty}\mathcal{H}_{V,A}((\lambda p)^2, (q-\lambda p)^2, q^2)$$
$$= -\frac{1}{\lambda^2}\langle\overline{\psi}\psi\rangle_0\frac{1}{p^2q^2} + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \quad (3.17)$$
$$\lim_{\lambda\to\infty}\mathcal{H}_V((\lambda p)^2, q^2, (q+\lambda p)^2)$$
$$= \frac{1}{\lambda^2}\frac{1}{p^2}\Pi_{VT}(q^2) + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \quad (3.18)$$

$$\lim_{\lambda\to\infty}\mathcal{H}_A((\lambda p)^2, q^2, (q+\lambda p)^2) = \mathcal{O}\left(\frac{1}{\lambda^3}\right). \quad (3.19)$$

4 The intermediate energy region

In this section, we shall construct representations of the invariant functions which describe the correlators $\langle VAP\rangle$, $\langle VVP\rangle$, $\langle AAP\rangle$ and $\langle VT\rangle$ in the intermediate energy region, dominated by the resonances, and which reproduce the short-distance constraints studied in the preceding section. Finding the general structure of the invariant functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}_{V,A}$ is of course far beyond our present possibilities. As discussed in the introduction, we shall therefore work in the framework of the large-N_C approximation to QCD and assume, in addition, that already a finite number of resonances will give a satisfactory description of these correlators.

We first consider the case where only a single resonance is retained in each channel, assuming that in the pseudoscalar one only the massless Goldstone bosons need to be kept. The corresponding lowest meson dominance (LMD) ansätze for the invariant functions are constructed such that the resulting expressions agree with the short-distance constraints (3.2)–(3.4). This is rather straightforward, and in the case of the $\langle VAP \rangle$ correlator, the result reads [33]

$$\begin{aligned}&\mathcal{F}^{\rm LMD}(p^2,q^2,(p+q)^2)\\&\quad=\frac{\langle\overline{\psi}\psi\rangle_0}{2}\frac{p^2-q^2-(p+q)^2+2a}{(p^2-M_V^2)(q^2-M_A^2)(p+q)^2},\\&\mathcal{G}^{\rm LMD}(p^2,q^2,(p+q)^2)\\&\quad=\langle\overline{\psi}\psi\rangle_0\frac{-q^2+b}{(p^2-M_V^2)(q^2-M_A^2)q^2(p+q)^2}.\end{aligned}\tag{4.1}$$

The constants a and b in (4.1) can be determined as follows. As shown in [33], one can relate the $\langle VA|\pi\rangle$ vertex function Γ_{VA} (for the definitions and a discussion of some properties of the vertex functions associated to the three-point functions under study, we refer the reader to Appendix A) to the two-point correlator $\langle VV\text{–}AA\rangle$ via a low-energy theorem. From the two Weinberg sum rules [40] one obtains in this way the relation

$$a-b=-(M_V^2+M_A^2).\tag{4.2}$$

The constant b can be fixed by requiring that the vector form factor of the pion $F_V^\pi(q^2)$, defined by

$$\langle\pi^a(p')|V_\mu^b(0)|\pi^c(p)\rangle=\mathrm{i}f^{abc}(p'+p)_\mu F_V^\pi(q^2),\qquad q=p-p',\tag{4.3}$$

satisfies an unsubtracted dispersion relation [20] (there are theoretical arguments [41,42] in favor of a $1/q^2$ fall-off of this form factor for large momentum transfer). Since

$$\begin{aligned}F_V^\pi(q^2)\equiv 1+&\frac{q^2}{2\langle\overline{\psi}\psi\rangle_0}\lim_{(q-p)^2\to0}\lim_{p^2\to0}(q-p)^2p^2\\&\times\mathcal{G}(q^2,(q-p)^2,p^2),\end{aligned}$$

we obtain with the function $\mathcal{G}^{\rm LMD}$ from (4.1) the result

$$F_V^{\pi,\rm LMD}(q^2)=1-\frac{b}{2M_A^2}\frac{q^2}{q^2-M_V^2},\tag{4.4}$$

and thus

$$b=2M_A^2.\tag{4.5}$$

Combining (4.2) and (4.5) then gives

$$a=M_A^2-M_V^2.\tag{4.6}$$

We note that the argument given in [33] that the same result for the constant b can also be obtained by enforcing the correct short-distance behavior of the $\langle VP|\pi\rangle$ vertex function Γ_{VP} is not correct. In Appendix A we sketch the derivation of the operator product expansion of this vertex function, with the result given in (A.6). The LMD ansatz will reproduce the subleading term in the OPE provided $b=4M_A^2/3$, which implies with (4.2) $a=-M_V^2+M_A^2/3$. Thus, with the LMD ansatz for the invariant function $\mathcal{G}$ from (4.1) it is not possible to reproduce at the same time the requirements from the asymptotic behavior of the form factor $F_V^\pi(q^2)$ and from the subleading terms in the OPE for the $\langle VP|\pi\rangle$ vertex function. The subleading term of the OPE of $\langle VA|\pi\rangle$ is correctly reproduced if (4.2) holds. In what follows, we shall understand that the LMD approximation for the $\langle VAP\rangle$ correlator corresponds to the ansatz (4.1) together with the choice $b=2M_A^2$.

For the $\langle VT\rangle$ two-point functions, only $J^{PC}=1^{--}$ vector mesons contribute. In the LMD approximation, the leading short-distance behavior then fixes everything but the mass of the lowest vector resonance,

$$\Pi_{VT}^{\rm LMD}(p^2)=-\langle\overline{\psi}\psi\rangle_0\frac{1}{p^2-M_V^2}.\tag{4.7}$$

It is quite remarkable that with this simple ansatz all the remaining leading short-distance constraints explicited in the previous section are met. In particular, for the quantities introduced in (3.13) we obtain

$$\mathcal{F}^{(0),\rm LMD}(q^2)=0,\ \ \mathcal{F}^{(1),\rm LMD}(q^2)=\frac{1}{q^2},$$
$$\mathcal{G}^{(0),\rm LMD}(q^2)=-\frac{1}{q^2},\ \ \mathcal{G}^{(1),\rm LMD}(q^2)=-\frac{2}{q^2},$$

which satisfy (3.14). Note, however, that the LMD ansätze for $\mathcal{F}$ and $\mathcal{G}$ in (4.1) do not correctly reproduce the subleading terms in the OPE for the correlator $\langle VAP\rangle$ given in [33].

The LMD ansatz for the invariant functions $\mathcal{H}_{V,A}$ reads[3]

$$\begin{aligned}&\mathcal{H}_V^{\rm LMD}(p^2,q^2,(p+q)^2)\\&\quad=-\frac{\langle\overline{\psi}\psi\rangle_0}{2}\frac{p^2+q^2+(p+q)^2-c_V}{(p^2-M_V^2)(q^2-M_V^2)(p+q)^2},\\&c_V=\frac{N_C}{4\pi^2}\frac{M_V^4}{F_0^2},\\&\mathcal{H}_A^{\rm LMD}(p^2,q^2,(p+q)^2)\\&\quad=-\frac{\langle\overline{\psi}\psi\rangle_0}{2}\frac{p^2+q^2-(p+q)^2-c_A}{(p^2-M_A^2)(q^2-M_A^2)(p+q)^2},\\&c_A=\frac{N_C}{12\pi^2}\frac{M_A^4}{F_0^2},\end{aligned}\tag{4.8}$$

where $c_{V,A}$ are fixed by the Wess–Zumino–Witten anomaly [44] term. Again, these simple ansätze fulfill all the remaining leading short-distance requirements worked out in the

[3] The LMD ansatz for the $\langle VVP\rangle$ three-point function was given in [32,43]

preceding section. We note that for the vertex functions Γ_{VV} and Γ_{AA} also the subleading terms in the OPE in (A.6) are reproduced by these ansätze.

As already pointed out, it is sometimes necessary to generalize the ansätze for the invariant functions given above by including more than one resonance in each channel. This might be due to some additional constraints that are imposed on the Green's functions or in order to better reproduce experimental data involving resonances, as was argued in [32]. If we include, for instance, one additional vector resonance, the expressions for the invariant functions $\mathcal{F}, \mathcal{G}, \Pi_{VT}$ and $\mathcal{H}_V$ read as follows ($\mathcal{H}_A$ remains, of course, unchanged)

$$\begin{aligned}
&\mathcal{F}^{\mathrm{LMD}+V}(p^2,q^2,(p+q)^2)\\
&= \frac{\langle\overline{\psi}\psi\rangle_0}{2}\frac{p^2\left[p^2-q^2-(p+q)^2\right]+P_F(p^2,q^2,(p+q)^2)}{(p^2-M_{V_1}^2)(p^2-M_{V_2}^2)(q^2-M_A^2)(p+q)^2},\\
&\mathcal{G}^{\mathrm{LMD}+V}(p^2,q^2,(p+q)^2)\\
&= \langle\overline{\psi}\psi\rangle_0\frac{-p^2q^2+P_G(p^2,q^2,(p+q)^2)}{(p^2-M_{V_1}^2)(p^2-M_{V_2}^2)(q^2-M_A^2)q^2(p+q)^2},\\
&\Pi_{VT}^{\mathrm{LMD}+V}(p^2) = -\langle\overline{\psi}\psi\rangle_0\frac{p^2+c_{VT}}{(p^2-M_{V_1}^2)(p^2-M_{V_2}^2)},\\
&\mathcal{H}_V^{\mathrm{LMD}+V}(p^2,q^2,(p+q)^2) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\\
&\times\frac{p^2q^2\left[p^2+q^2+(p+q)^2\right]+P_H^V(p^2,q^2,(p+q)^2)}{(p^2-M_{V_1}^2)(p^2-M_{V_2}^2)(q^2-M_{V_1}^2)(q^2-M_{V_2}^2)(p+q)^2},
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
&P_F(p^2,q^2,(p+q)^2) = f_1p^2+f_2q^2+f_3(p+q)^2+f_4,\\
&P_G(p^2,q^2,(p+q)^2) = g_1p^2+g_2q^2+g_3(p+q)^2+g_4,\\
&P_H^V(p^2,q^2,(p+q)^2) = h_1(p^2+q^2)^2+h_2p^2q^2\\
&\quad +h_3(p^2+q^2)(p+q)^2+h_4(p+q)^4+h_5(p^2+q^2)\\
&\quad +h_6(p+q)^2+h_7.
\end{aligned}$$

The coefficients that appear in the polynomials P_F, P_G and P_H^V have to fulfill the following relations

$$\begin{aligned}
f_2+f_3 &= -2c_{VT},\\
h_1+h_3+h_4 &= 2c_{VT},
\end{aligned} \tag{4.10}$$

in order to reproduce all short-distance constraints from the operator product expansion given in the previous section. Furthermore, the Wess–Zumino–Witten anomaly determines

$$h_7 = -\frac{N_C}{4\pi^2}\frac{M_{V_1}^4M_{V_2}^4}{F_0^2}.$$

Using the low-energy theorem that relates Γ_{VA} to $\langle VV-AA\rangle$ one obtains, from the LMD+V ansatz for the latter correlator [45], the relations

$$\begin{aligned}
f_1+f_2 &= 2\left(g_1+g_2-M_{V_1}^2-M_{V_2}^2-M_A^2\right),\\
f_4 &= 2\biggl(g_4+M_{V_1}^2M_{V_2}^2+M_A^2(M_{V_1}^2+M_{V_2}^2)\\
&\quad -\frac{4\pi\alpha_s\langle\overline{\psi}\psi\rangle_0^2}{F_0^2}\biggr).
\end{aligned} \tag{4.11}$$

The first of these conditions also guarantees that the next-to-leading short-distance behavior of Γ_{VA} is correctly reproduced. The vector form factor of the pion now reads

$$F_V^{\pi,\mathrm{LMD}+V}(q^2) = 1-\frac{q^2}{2M_A^2}\frac{g_1q^2+g_4}{(q^2-M_{V_1}^2)(q^2-M_{V_2}^2)}. \tag{4.12}$$

Requiring that it behaves like $1/q^2$ for large q^2, leads to the relation

$$g_1 = 2M_A^2. \tag{4.13}$$

On the other hand, in order to reproduce the subleading terms in the OPE for Γ_{VP}, (A.6), with the LMD+V ansatz for $\mathcal{G}$ yields a constraint which is independent from the previous one,

$$g_1+g_3 = \frac{4}{3}M_A^2. \tag{4.14}$$

Combining this equation with (4.13), we obtain the result

$$g_3 = -\frac{2}{3}M_A^2.$$

Note that in contrast to the LMD case above, we can simultaneously fulfill the requirements from the asymptotic behavior of the form factor $F_V^\pi(q^2)$ and from the subleading terms in the OPE for the Γ_{VP} vertex functions.

Finally we note that the subleading terms in the OPE for the $\langle VV|\pi\rangle$ vertex function Γ_{VV} in (A.6) are reproduced by the LMD+V ansatz for $\mathcal{H}_V$ without leading to further constraints on the coefficients h_i.

Let us briefly mention the ansatz for $\mathcal{H}_V$ with one additional pseudoscalar resonance, discussed in [32],

$$\begin{aligned}
&\mathcal{H}_V^{\mathrm{LMD}+P}(p^2,q^2,(p+q)^2) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\\
&\times\frac{(p+q)^2\left[p^2+q^2+(p+q)^2\right]+P_H^P(p^2,q^2,(p+q)^2)}{(p^2-M_V^2)(q^2-M_V^2)((p+q)^2-M_P^2)(p+q)^2},
\end{aligned} \tag{4.15}$$

where

$$P_H^P(p^2,q^2,(p+q)^2) = \hat{h}_1(p^2+q^2)+\hat{h}_2(p+q)^2+\hat{h}_3.$$

The OPE leads to the condition

$$\hat{h}_1 = -M_P^2, \tag{4.16}$$

and the Wess–Zumino–Witten anomaly yields

$$\hat{h}_3 = \frac{N_C}{4\pi^2}\frac{M_V^4M_P^2}{F_0^2}.$$

In contrast to the LMD case, the coefficients that appear in the invariant functions are no longer fixed unambiguously by the leading terms in the OPE. Here we have considered additional restrictions arising from the next-to-leading short-distance behavior of the $\langle VA|\pi\rangle$, $\langle VP|\pi\rangle$

and $\langle VV|\pi\rangle$ vertex functions. Further information may be gained from the study of vertex functions like $\langle VP|a_1\rangle$, $\langle VV|\rho\rangle$, etc. We shall not pursue this interesting line of thought here. Other sources of additional constraints might also be invoked, either from low-energy physics, from subleading terms in the OPE of the three-point functions[4] or from processes involving resonances. We shall illustrate this point below.

5 Determination of low-energy constants

In this section, we shall use the ansätze of the previous section in order to obtain an evaluation of the low-energy constants involved in the chiral expansion of the three correlators under study. This is done upon performing the low-energy expansion of the invariant functions $\mathcal{F}^{\rm LMD}$, $\mathcal{G}^{\rm LMD}$ and $\mathcal{H}^{\rm LMD}_{V,A}$ and by subsequently matching it with the chiral expressions (2.5) and (2.6).

We start with the sector of even intrinsic parity, i.e. with the $\langle VAP\rangle$ correlator. For small momentum transfers, we can expand the resonance propagators

$$\frac{1}{p^2-M_{\rm R}^2} = -\frac{1}{M_{\rm R}^2}\left[1+\mathcal{O}\left(\frac{p^2}{M_{\rm R}^2}\right)\right], \qquad (5.1)$$

and obtain (in these expressions, we have taken $b=2M_A^2$, $a=M_A^2-M_V^2$; see the discussion after (4.1)),

$$\begin{aligned}\mathcal{F}^{\rm LMD}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{(p+q)^2}\\ &\times\left[\frac{1}{M_V^2}-\frac{1}{M_A^2}+\frac{p^2}{M_V^2M_A^2}\left(\frac{M_A^2}{M_V^2}-\frac{1}{2}\right)\right.\\ &\left.+\frac{q^2}{M_V^2M_A^2}\left(\frac{1}{2}-\frac{M_V^2}{M_A^2}\right)-\frac{1}{2}\frac{(p+q)^2}{M_V^2M_A^2}+\mathcal{O}(p^4)\right],\end{aligned}$$

$$\begin{aligned}\mathcal{G}^{\rm LMD}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{q^2(p+q)^2}\\ &\times\left[\frac{2}{M_V^2}+p^2\left(\frac{2}{M_V^4}\right)+q^2\left(\frac{1}{M_V^2M_A^2}\right)+\mathcal{O}(p^4)\right],\end{aligned}$$

where $\mathcal{O}(p^4)$ includes all possible higher order polynomials in $p^2, q^2, (p+q)^2$. Comparison with the expressions (2.5) from ChPT yields the following solution (treating $SU(2)$ and $SU(3)$ together)

$$\begin{aligned}L_9^{\rm LMD} &= -\frac{1}{2}l_6^{\rm LMD} = \frac{1}{2}\frac{F_0^2}{M_V^2},\\ L_{10}^{\rm LMD} &= l_5^{\rm LMD} = -\frac{1}{4}\frac{F_0^2}{M_V^2}-\frac{1}{4}\frac{F_0^2}{M_A^2},\end{aligned} \qquad (5.2)$$

and

$$\begin{aligned}C_{78}^{\rm LMD} &= c_{44}^{\rm LMD} = \frac{3}{8}\frac{F_0^2}{M_V^4}+\frac{3}{8}\frac{F_0^2}{M_V^2M_A^2},\\ C_{82}^{\rm LMD} &= c_{47}^{\rm LMD} = -\frac{1}{8}\frac{F_0^2}{M_V^4}-\frac{1}{8}\frac{F_0^2}{M_V^2M_A^2},\\ C_{87}^{\rm LMD} &= c_{50}^{\rm LMD} = \frac{1}{8}\frac{F_0^2}{M_V^4}+\frac{1}{8}\frac{F_0^2}{M_V^2M_A^2}+\frac{1}{8}\frac{F_0^2}{M_A^4},\\ C_{88}^{\rm LMD} &= c_{51}^{\rm LMD} = -\frac{1}{4}\frac{F_0^2}{M_V^4},\\ C_{89}^{\rm LMD} &= c_{52}^{\rm LMD} = \frac{3}{4}\frac{F_0^2}{M_V^4}+\frac{1}{2}\frac{F_0^2}{M_V^2M_A^2},\\ C_{90}^{\rm LMD} &= c_{53}^{\rm LMD} = 0.\end{aligned} \qquad (5.3)$$

The results for L_9 and L_{10} agree with those obtained in [18, 20] after employing the two Weinberg sum rules [40] and using, in addition, the relation $F_VG_V=F_0^2$ which follows from the assumption that the vector form factor of the pion $F_V^\pi(q^2)$ satisfies an unsubtracted dispersion relation [20].

Going through the same steps for the two correlators $\langle VVP\rangle$ and $\langle AAP\rangle$, we obtain the small momentum expansions

$$\begin{aligned}\mathcal{H}_V^{\rm LMD}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{(p+q)^2}\\ &\times\left[\frac{N_C}{8\pi^2F_0^2}-\frac{p^2+q^2}{M_V^4}\left(\frac{1}{2}-\frac{N_C}{8\pi^2}\frac{M_V^2}{F_0^2}\right)-\frac{(p+q)^2}{M_V^4}\right.\\ &\left.+\mathcal{O}(p^4)\right],\\ \mathcal{H}_A^{\rm LMD}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{(p+q)^2}\\ &\times\left[\frac{N_C}{24\pi^2F_0^2}-\frac{p^2+q^2}{M_A^4}\left(\frac{1}{2}-\frac{N_C}{24\pi^2}\frac{M_A^2}{F_0^2}\right)+\frac{(p+q)^2}{M_A^4}\right.\\ &\left.+\mathcal{O}(p^4)\right],\end{aligned}$$

from which we infer the following equations for some of the $\mathcal{O}(p^6)$ low-energy constants A_i of [5] in the odd intrinsic parity sector, see (2.6),

$$\begin{aligned}A_2^{\rm LMD}-4A_3^{\rm LMD} &= \frac{F_0^2}{8M_V^4}-\frac{N_C}{32\pi^2}\frac{1}{M_V^2},\\ A_2^{\rm LMD}-2A_3^{\rm LMD}-4A_4^{\rm LMD} &= -\frac{F_0^2}{8M_V^4},\\ A_{11}^{\rm LMD}+A_{23}^{\rm LMD}-4A_{24}^{\rm LMD} &= \frac{F_0^2}{8M_A^4}-\frac{N_C}{96\pi^2}\frac{1}{M_A^2},\\ 3A_{11}^{\rm LMD}+4A_{17}^{\rm LMD}+A_{23}^{\rm LMD}-2A_{24}^{\rm LMD}-4A_{25}^{\rm LMD} &= \frac{F_0^2}{8M_A^4}.\end{aligned} \qquad (5.4)$$

The numerical values that follow from these expressions for the low-energy constants C_i and A_i are discussed in the next section.

[4] Actually, the addition of a single vector or pseudoscalar resonance is still not sufficient in order to reproduce the next-to-leading short-distance behavior of $\langle VAP\rangle$ as given by (46) of [33]

6 Comparison with the resonance Lagrangian approach

In this section, we wish to compare the preceding determination of the low-energy constants with the approach which uses a Lagrangian with explicit resonance degrees of freedom. For definiteness, we use the resonance Lagrangian given in [38] which has often been employed in the literature to estimate the low-energy constants at order p^6. In this Lagrangian a vector-field representation is used for the vector and axial-vector resonances. For convenience, we have written down this Lagrangian in Appendix B. Note that no pseudoscalar resonances appear in this Lagrangian. Furthermore, as stressed in [20], one has to add local terms from $\mathcal{L}_4$, [5] with fixed coefficients $L_i^{\rm res}$ in order to correctly reproduce the QCD short-distance behavior of *certain* Green's functions. We shall come back to this point below.

The calculation of the $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$ three-point functions with the resonance Lagrangian yields again a result in agreement with the general solution of the Ward identities with the following expressions for the invariant functions (the coupling β_V is sometimes denoted by f_χ; see e.g. [46])

$$\begin{aligned}\mathcal{F}^{\rm res}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2}\\ &\times\Bigg[4L_9^{\rm res}+4L_{10}^{\rm res}+\frac{p^2}{(p^2-M_V^2)}\\ &\times\left(f_V^2-2f_Vg_V+2\sqrt{2}f_V\alpha_V\right)\\ &+\frac{q^2}{(q^2-M_A^2)}\left(-f_A^2-2\sqrt{2}f_A\alpha_A\right)\\ &+\frac{p^2q^2}{(p^2-M_V^2)(q^2-M_A^2)}\\ &\times\left(-2f_Vf_A(A^{(2)}-A^{(3)})\right)\Bigg], \end{aligned}\qquad(6.1)$$

$$\begin{aligned}\mathcal{G}^{\rm res}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2q^2(p+q)^2}\\ &\times\Bigg[4L_9^{\rm res}+\frac{1}{(p^2-M_V^2)}\\ &\times\left(-2f_Vg_Vp^2+2\sqrt{2}f_V\alpha_Vq^2-4\sqrt{2}f_V\beta_V(p+q)^2\right)\\ &+\frac{q^2}{(q^2-M_A^2)}\left(-2\sqrt{2}f_A\alpha_A\right)\\ &+\frac{(-2f_Vf_A)q^2}{(p^2-M_V^2)(q^2-M_A^2)}\\ &\times\left(A^{(2)}q^2-A^{(3)}p^2+2B(p+q)^2\right)\Bigg], \end{aligned}\qquad(6.2)$$

and

$$\begin{aligned}\mathcal{H}_V^{res}(p^2,q^2,(p+q)^2) &= -\frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2}\\ &\times\Bigg[-\frac{N_C}{8\pi^2}+\frac{p^2}{(p^2-M_V^2)}\left(4\sqrt{2}f_Vh_V\right)\\ &+\frac{q^2}{(q^2-M_V^2)}\left(4\sqrt{2}f_Vh_V\right)\\ &-\frac{p^2q^2}{(p^2-M_V^2)(q^2-M_V^2)}\left(4f_V^2\sigma_V\right)\Bigg],\\ \mathcal{H}_A^{res}(p^2,q^2,(p+q)^2) &= -\frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2}\\ &\times\Bigg[-\frac{N_C}{24\pi^2}+\frac{p^2}{(p^2-M_A^2)}\left(4\sqrt{2}f_Ah_A\right)\\ &+\frac{q^2}{(q^2-M_A^2)}\left(4\sqrt{2}f_Ah_A\right)\\ &-\frac{p^2q^2}{(p^2-M_A^2)(q^2-M_A^2)}\left(4f_A^2\sigma_A\right)\Bigg]. \end{aligned}\qquad(6.3)$$

For momentum transfers that are small as compared to the resonance masses, we can expand the propagators as sketched in (5.1). Therefore, the contributions from the resonance Lagrangian start at $\mathcal{O}(p^6)$ in the chiral expansion. The contributions $A^{(2)}, A^{(3)}, B$, and σ_V, σ_A originate from the exchange of two resonances and start to contribute to the low-energy expansion at $\mathcal{O}(p^8)$ only.

6.1 Even intrinsic parity sector

For the $\langle VAP\rangle$ correlator, we thus find the low-energy expansions

$$\begin{aligned}\mathcal{F}^{\rm res}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2(p+q)^2}\\ &\times\Bigg[4L_9^{\rm res}+4L_{10}^{\rm res}-\frac{p^2}{M_V^2}\left(f_V^2-2f_Vg_V+2\sqrt{2}f_V\alpha_V\right)\\ &-\frac{q^2}{M_A^2}\left(-f_A^2-2\sqrt{2}f_A\alpha_A\right)+\mathcal{O}(p^4)\Bigg],\\ \mathcal{G}^{\rm res}(p^2,q^2,(p+q)^2) &= \frac{\langle\overline{\psi}\psi\rangle_0}{F_0^2q^2(p+q)^2}\Bigg[4L_9^{\rm res}-\frac{1}{M_V^2}\\ &\times\left(-2f_Vg_Vp^2+2\sqrt{2}f_V\alpha_Vq^2-4\sqrt{2}f_V\beta_V(p+q)^2\right)\\ &-\frac{q^2}{M_A^2}\left(-2\sqrt{2}f_A\alpha_A\right)+\mathcal{O}(p^4)\Bigg]. \end{aligned}$$

Comparison with the expressions of the functions $\mathcal{F}$ and $\mathcal{G}$ in ChPT, (2.5), leads to the following determination of the corresponding low-energy constants (again treating $SU(2)$ and $SU(3)$ together)

$$\begin{aligned}C_{78}^{\rm res} = c_{44}^{\rm res} &= \frac{1}{4}\frac{1}{M_V^2}f_V^2+\frac{1}{8}\frac{1}{M_V^2}f_Vg_V+\frac{1}{2\sqrt{2}}\frac{1}{M_V^2}f_V\beta_V\\ &+\frac{1}{\sqrt{2}}\frac{1}{M_A^2}f_A\alpha_A, \end{aligned}$$

[5] Where $\mathcal{L}_n=\sum_{k+2l=n}\mathcal{L}_{(k,l)}$ in terms of the notation introduced in (1.1)

Table 1. Numerical values for the low-energy constants C_i (in units of $10^{-4}/F_0^2$) obtained from the LMD estimates in (5.3) and with the expressions derived from the resonance Lagrangian in (6.4) for two different sets of input values for the resonance parameters

	C_{78}	C_{82}	C_{87}	C_{88}	C_{89}	C_{90}
LMD	1.09	-0.36	0.40	-0.52	1.97	0.0
Set I	1.09	-0.29	0.47	-0.16	2.29	0.33
Set II	1.49	-0.39	0.65	-0.14	3.22	0.51

$$
\begin{aligned}
C_{82}^{\rm res} &= c_{47}^{\rm res} = -\frac{1}{16}\frac{1}{M_V^2} f_V^2 - \frac{1}{16}\frac{1}{M_V^2} f_V g_V \\
&\quad - \frac{1}{4\sqrt{2}}\frac{1}{M_V^2} f_V \beta_V - \frac{1}{4\sqrt{2}}\frac{1}{M_A^2} f_A \alpha_A, \\
C_{87}^{\rm res} &= c_{50}^{\rm res} = \frac{1}{8}\frac{1}{M_V^2} f_V^2 - \frac{1}{8}\frac{1}{M_A^2} f_A^2, \\
C_{88}^{\rm res} &= c_{51}^{\rm res} = -\frac{1}{4}\frac{1}{M_V^2} f_V g_V - \frac{1}{\sqrt{2}}\frac{1}{M_V^2} f_V \beta_V, \\
C_{89}^{\rm res} &= c_{52}^{\rm res} = \frac{1}{2}\frac{1}{M_V^2} f_V^2 + \frac{1}{4}\frac{1}{M_V^2} f_V g_V + \frac{1}{\sqrt{2}}\frac{1}{M_V^2} f_V \alpha_V \\
&\quad + \frac{1}{\sqrt{2}}\frac{1}{M_A^2} f_A \alpha_A, \\
C_{90}^{\rm res} &= c_{53}^{\rm res} = -\frac{1}{\sqrt{2}}\frac{1}{M_V^2} f_V \beta_V . \qquad (6.4)
\end{aligned}
$$

There are two main differences with the LMD ansatz of the previous section: the absence of a term $(p+q)^2$ in the low-momentum expansion of $\mathcal{F}^{\rm res}(p^2, q^2, (p+q)^2)$, whereas such a term is present in $\mathcal{G}^{\rm res}(p^2, q^2, (p+q)^2)$ but not in $\mathcal{G}^{\rm LMD}(p^2, q^2, (p+q)^2)$.

In Table 1 (see e.g. (5.3) for the translation into the corresponding $SU(2)$ constants c_i) we compare the numerical values for the low-energy constants C_i in the LMD approximation with those obtained from the resonance Lagrangian. As recalled in the introduction and as discussed in [18], these numbers have to be understood as the values of the low-energy constants at the scale set by Λ_H, which we identify with the ρ mass M_V. We have used the values $F_0 = 92.4\,\mathrm{MeV}$, $M_V = 769\,\mathrm{MeV}$, and $M_A = 1230\,\mathrm{MeV}$, as well as the two sets of input values for the resonance parameters listed in Table 2 as given in [38] (Set I), based on an ENJL model, and from [46,47] (Set II), extracted from resonance decays. If we allow for a relative error of about 30% in these values, a typical size for the uncertainty attached to a large-N_C estimate, the agreement is rather good, except for the cases of C_{88} and C_{90}. We postpone the discussion of some phenomenological implications of the differences shown by Table 1 to Sect. 8 below.

The values displayed in the second line of Table 1 were obtained by taking $b = 2M_A^2$ in the LMD ansatz (4.1). If we had taken $b = 4M_A^2/3$ instead, these values would have changed within an acceptable range: about 20% for C_{78} and C_{82}, 16% in the case of C_{89}. The biggest variation occurs for C_{88}, around 30%, while C_{87} and C_{90} remain unchanged, being insensitive to the value of b. Notice also that L_9 is proportional to b, $L_9 = F_0^2 b/4M_V^2 M_A^2$. Changing b from $2M_A^2$ to $4M_A^2/3$ decreases the value of L_9 by as much as 33%, while leaving L_{10} unaffected. This modifies the $\mathcal{O}(p^4)$ prediction of the pion charge radius from $\langle r^2\rangle_V^\pi(b = 2M_A^2) = 0.47 \pm 0.13\,\mathrm{fm}^2$ to $\langle r^2\rangle_V^\pi(b = 4M_A^2/3) = 0.33 \pm 0.09\,\mathrm{fm}^2$, somewhat lower than the experimental value $\langle r^2\rangle_V^\pi = 0.439 \pm 0.008\,\mathrm{fm}^2$ [48].

Table 2. Values for the parameters in the resonance Lagrangian from [38] (Set I) and from [46,47] (Set II)

	f_V	g_V	α_V	$\beta_V \equiv f_\chi$	f_A	α_A
Set I	0.17	0.08	-0.015	-0.019	0.085	-0.0092
Set II	0.20	0.09	-0.014	-0.025	0.10	-0.0067

At this stage, it is worthwhile to stress that it is not possible to find a one-to-one correspondence between the parameter sets of the resonance Lagrangian and of the LMD approximation to large-N_C QCD. This directly reflects the differences mentioned above in the low-energy expansions of the functions $\mathcal{F}^{\rm LMD}$ and $\mathcal{G}^{\rm LMD}$ on the one hand, and of $\mathcal{F}^{\rm res}$ and $\mathcal{G}^{\rm res}$ on the other hand. Nevertheless, some agreement can be obtained by adjusting the resonance parameters. When performing such a parametric comparison, one has to observe that the LMD ansatz already encodes some additional information. For instance, the two Weinberg sum rules [40] are fulfilled. In the LMD approximation they take the form $F_V^2 = F_0^2 + F_A^2$ and $F_V^2 M_V^2 = F_A^2 M_A^2$ and allow one to express F_V and F_A through the resonance masses and F_0. Furthermore, in the LMD approximation the identity $F_V G_V = F_0^2$ holds. Finally, one has to make the following identifications between the parameters in the vector- and the tensor-field representation for the resonances: $f_V \equiv F_V/M_V, g_V \equiv G_V/M_V$ and $f_A \equiv F_A/M_A$; see [20].

Comparing the expressions for C_{90} and C_{88} in the LMD approximation, (5.3), with those obtained from the resonance Lagrangian, (6.4), and using $F_V G_V = F_0^2$ we get

$$\beta_V = 0.$$

This removes the largest numerical discrepancies in Table 1. Note in particular the huge cancellation in $C_{88}^{\rm res}$ for the values of β_V given in Table 2. On the other hand, using the Weinberg sum rules and the identifications mentioned above one notices that

$$C_{87}^{\rm LMD} \equiv C_{87}^{\rm res}.$$

Thus we are left with the three equations $C_i^{\rm LMD} = C_i^{\rm res}$, $i = 78, 82, 89$, for the remaining two unknowns α_V and α_A. It turns out that this system of equations is inconsistent. We can solve, however, for α_V, since in the difference $C_{78} - C_{89}$ the term with $f_A \alpha_A$ drops out. From the requirement $C_{78}^{\rm LMD} - C_{89}^{\rm LMD} = C_{78}^{\rm res} - C_{89}^{\rm res}$ we obtain

$$\alpha_V = -\frac{\sqrt{2}}{8}\frac{F_0 M_V}{M_A^3}\frac{(M_A^2 + M_V^2)}{\sqrt{M_A^2 - M_V^2}} = -0.015,$$

in remarkable agreement with the values quoted in [38, 46,47], see Table 2. On the other hand, requiring $C_{78}^{\mathrm{LMD}} + 4C_{82}^{\mathrm{LMD}} = C_{78}^{\mathrm{res}} + 4C_{82}^{\mathrm{res}}$, leads to

$$-\frac{1}{8}\frac{F_0^2}{M_V^2 M_A^2} = 0.$$

This is not compatible with the spontaneous breakdown of chiral symmetry [1] and the Goldstone theorem, which requires $F_0 \neq 0$.

6.2 Odd intrinsic parity sector

The low-energy expansion of the resonance expressions for the two correlators $\langle VVP \rangle$ and $\langle AAP \rangle$ gives the following estimates of the low-energy constants A_i

$$\begin{aligned} A_2^{\mathrm{res}} - 4A_3^{\mathrm{res}} &= -\sqrt{2} f_V h_V \frac{1}{M_V^2}, \\ A_2^{\mathrm{res}} - 2A_3^{\mathrm{res}} - 4A_4^{\mathrm{res}} &= 0, \\ A_{11}^{\mathrm{res}} + A_{23}^{\mathrm{res}} - 4A_{24}^{\mathrm{res}} &= -\sqrt{2} f_A h_A \frac{1}{M_A^2}, \\ 3A_{11}^{\mathrm{res}} + 4A_{17}^{\mathrm{res}} + A_{23}^{\mathrm{res}} - 2A_{24}^{\mathrm{res}} - 4A_{25}^{\mathrm{res}} &= 0. \end{aligned} \quad (6.5)$$

In Table 3 we compare the numerical values for the low-energy constants A_i from the LMD estimates in (5.4) with those from the resonance Lagrangian, (6.5). We have introduced the following notations for the combinations of low-energy constants that appear in $\langle VVP \rangle$ and $\langle AAP \rangle$

$$\begin{aligned} A_{V,p^2} &= A_2 - 4A_3, \\ A_{V,(p+q)^2} &= A_2 - 2A_3 - 4A_4, \\ A_{A,p^2} &= A_{11} + A_{23} - 4A_{24}, \\ A_{A,(p+q)^2} &= 3A_{11} + 4A_{17} + A_{23} - 2A_{24} - 4A_{25}. \end{aligned} \quad (6.6)$$

The agreement for the low-energy constants A_{V,A,p^2} is quite good, whereas the two approaches give different results for $A_{V,A,(p+q)^2}$. In the expressions for the low-energy constants A_i from the resonance Lagrangian (6.5) we used the ENJL estimates (Set I)

$$\begin{aligned} f_V h_V &= \frac{N_C}{16\pi^2}\frac{\sqrt{2}}{8}(1+g_A) = 0.0055, \\ f_A h_A &= \frac{N_C}{16\pi^2}\frac{\sqrt{2}}{24} g_A(1+g_A) = 0.0012, \end{aligned} \quad (6.7)$$

with $N_C = 3$ and $g_A = 0.65$; see [38], in particular the Erratum. We shall discuss estimates for the low-energy constants A_{V,p^2} and $A_{V,(p+q)^2}$ beyond the LMD approximation in Sect. 8.3.

Again, it is impossible to find a one-to-one relation between the parameters of the resonance Lagrangian and those that describe the LMD ansatz. The reason lies in the absence of a term proportional to $(p+q)^2$ in the low-energy expansions of both $\mathcal{H}_V^{\mathrm{res}}$ and $\mathcal{H}_A^{\mathrm{res}}$. We notice nevertheless that requiring $A_{V,p^2}^{\mathrm{LMD}} = A_{V,p^2}^{\mathrm{res}}$ and $A_{A,p^2}^{\mathrm{LMD}} = A_{A,p^2}^{\mathrm{res}}$ leads to

Table 3. Numerical values for the combinations of low-energy constants A_i defined in (6.6), in units of $10^{-4}/F_0^2$, obtained from the LMD estimates in (5.4) and from the resonance Lagrangian in (6.5) (Set I)

	A_{V,p^2}	$A_{V,(p+q)^2}$	A_{A,p^2}	$A_{A,(p+q)^2}$
LMD	-1.11	-0.26	-0.14	0.040
Set I	-1.13	0.0	-0.096	0.0

the relations

$$\begin{aligned} f_V h_V &= \frac{1}{\sqrt{2}}\left(\frac{N_C}{32\pi^2} - \frac{F_0^2}{8M_V^2}\right) = 0.0054, \\ f_A h_A &= \frac{1}{\sqrt{2}}\left(\frac{N_C}{96\pi^2} - \frac{F_0^2}{8M_A^2}\right) = 0.0017, \end{aligned}$$

in rather good agreement with (6.7).

We conclude that, although an adequate adjustment of the parameters of the resonance Lagrangian of [38] can bring the determinations of the low-energy constants considered here to a reasonable numerical agreement, there is no way to establish an algebraic equivalence between the two approaches. We have focused here on the Lagrangian of [38] which is the most complete as far as couplings among resonances and to the Goldstone bosons are concerned. Therefore similar conclusions will also hold for other existing Lagrangians with resonance fields, see [14–20] and references therein.

7 QCD short-distance constraints on the resonance Lagrangian

The discrepancies between the estimates for the low-energy constants at $\mathcal{O}(p^6)$ from the LMD ansatz and from the resonance Lagrangian, as observed in the previous section, can be traced back to the different high-energy behaviors of the corresponding Green's functions in the two approaches. In fact, as we shall show in this section, the Green's function derived from the resonance Lagrangian are incompatible with the QCD short-distance constraints.

7.1 Two large momenta

We first consider the limit when the two momenta become simultaneously large, see (3.2)–(3.4). The invariant function $\mathcal{F}^{\mathrm{res}}$ from (6.1) behaves as

$$\begin{aligned} \lim_{\lambda\to\infty} \mathcal{F}^{\mathrm{res}}((\lambda p)^2, (\lambda q)^2, (\lambda p + \lambda q)^2) &= \frac{\langle \overline{\psi}\psi \rangle_0}{\lambda^2 F_0^2 (p+q)^2} \\ &\times [4L_9 + 4L_{10} + r_{FV} + r_{FA} + r_{FVA}] \\ &+ \frac{\langle \overline{\psi}\psi \rangle_0}{\lambda^4 F_0^2}\left[\frac{(r_{FV} + r_{FVA})M_V^2}{p^2(p+q)^2} + \frac{(r_{FA} + r_{FVA})M_A^2}{q^2(p+q)^2}\right] \\ &+ \mathcal{O}\left(\frac{1}{\lambda^6}\right), \end{aligned} \quad (7.1)$$

where r_{FV}, r_{FA}, and r_{FVA} denote the coefficients of the terms $p^2/(p^2-M_V^2), q^2/(q^2-M_A^2)$, and $(p^2q^2)/((p^2-M_V^2)(q^2-M_A^2))$ in (6.1), respectively. Agreement with the OPE result from (3.2) at order $1/\lambda^2$ can be achieved if the constraint

$$4L_9+4L_{10}+f_V^2-2f_Vg_V+2\sqrt{2}f_V\alpha_V-f_A^2 \\ -2\sqrt{2}f_A\alpha_A-2f_Vf_A(A^{(2)}-A^{(3)})=0 \qquad (7.2)$$

is fulfilled. However, at order $1/\lambda^4$, we observe that $\mathcal{F}^{\text{res}}$ is not compatible with the OPE result in (3.2), since the term $\sim 1/(p^2q^2)$ is missing in (7.1).

As was shown in [20], requiring agreement of certain Green's functions with the short-distance properties of QCD uniquely determines the low-energy constants L_i at $\mathcal{O}(p^4)$, even though the contributions of the resonance Lagrangian in the vector-field representation only start at $\mathcal{O}(p^6)$.[6] In fact, one has to add local terms from $\mathcal{L}_4$ to the resonance Lagrangian in order to obtain the correct short-distance behavior of these Green's functions. Analogously, at $\mathcal{O}(p^6)$, one might try to add local counterterms from $\mathcal{L}_6$. This is, however, not enough to bring the function $\mathcal{F}^{\text{res}}$ in agreement with the OPE. One has as well to add new terms involving both resonance fields and additional derivatives. A similar observation, concerning the $\langle VVS\rangle$ correlator, was made in [31]: the short-distance behavior of the three-point function $\langle VVS\rangle$ becomes consistent with the OPE only if a term $\langle(\hat{V}_{\mu\nu}-(f_V/2)f^+_{\mu\nu})^2\nabla^2\hat{S}\rangle$ is added to the resonance Lagrangian. In the present case the situation is more involved, i.e. several new terms would have to be added. We have not undertaken the task to construct them explicitly. One can show, however, that if one matches $\mathcal{F}^{\text{res}}$ with the constraints imposed by the OPE, the local counterterms at $\mathcal{O}(p^6)$ have to be adjusted in such a way that one finally obtains the same values for the low-energy constants as with the LMD approach. We caution the reader that the mere fact of using a tensor-field representation for the resonances does not, by itself, guarantee to yield the same estimates for the low-energy constants as with the LMD ansatz. Actually, it was pointed out in [33] that the resonance Lagrangian with a tensor-field representation leads to a correlator $\langle VAP\rangle$ [49] which does not have the correct short-distance properties.

We may perform a similar analysis for $\mathcal{G}^{\text{res}}$ from (6.2). In this case, one may obtain agreement with the QCD result of (3.3), provided the coupling constants in the resonance Lagrangian are adjusted as follows:

$$f_Vg_VM_V^2+\sqrt{2}f_V\alpha_VM_A^2-\sqrt{2}f_A\alpha_AM_V^2=\frac{F_0^2}{2}, \\ L_9=\frac{1}{2}f_Vg_V, \quad A^{(2)}=\sqrt{2}\frac{\alpha_V}{f_A}, \\ A^{(3)}=\sqrt{2}\frac{\alpha_A}{f_V}, \quad \beta_V=0, \quad B=0. \qquad (7.3)$$

We note, however, that in deriving these constraints we have not taken into account possible contributions to $\mathcal{G}^{\text{res}}$ from the new local terms that have to be added to the resonance Lagrangian in order to make $\mathcal{F}^{\text{res}}$ compatible with the OPE. The expression for L_9 in (7.3) agrees with [18,20], whereas the relations for $A^{(2)}$ and $A^{(3)}$ agree with those given in [38].

The short-distance behavior of the functions $\mathcal{H}_V^{\text{res}}$ and $\mathcal{H}_A^{\text{res}}$ in (6.3) in the odd intrinsic parity sector is given by

$$\lim_{\lambda\to\infty}\mathcal{H}_V^{\text{res}}((\lambda p)^2,(\lambda q)^2,(\lambda p+\lambda q)^2)=-\frac{\langle\overline{\psi}\psi\rangle_0}{\lambda^2F_0^2(p+q)^2} \\ \times\left[-\frac{N_C}{8\pi^2}+8\sqrt{2}f_Vh_V-4f_V^2\sigma_V\right]-\frac{\langle\overline{\psi}\psi\rangle_0M_V^2}{\lambda^4F_0^2} \\ \times\left[\frac{4\sqrt{2}f_Vh_V-4f_V^2\sigma_V}{p^2(p+q)^2}+\frac{4\sqrt{2}f_Vh_V-4f_V^2\sigma_V}{q^2(p+q)^2}\right] \\ +\mathcal{O}\left(\frac{1}{\lambda^6}\right),$$

$$\lim_{\lambda\to\infty}\mathcal{H}_A^{\text{res}}((\lambda p)^2,(\lambda q)^2,(\lambda p+\lambda q)^2)=-\frac{\langle\overline{\psi}\psi\rangle_0}{\lambda^2F_0^2(p+q)^2} \\ \times\left[-\frac{N_C}{24\pi^2}+8\sqrt{2}f_Ah_A-4f_A^2\sigma_A\right]-\frac{\langle\overline{\psi}\psi\rangle_0M_A^2}{\lambda^4F_0^2} \\ \times\left[\frac{4\sqrt{2}f_Ah_A-4f_A^2\sigma_A}{p^2(p+q)^2}+\frac{4\sqrt{2}f_Ah_A-4f_A^2\sigma_A}{q^2(p+q)^2}\right] \\ +\mathcal{O}\left(\frac{1}{\lambda^6}\right). \qquad (7.4)$$

Comparison of these expressions with the OPE result of (3.4) leads, at order $1/\lambda^2$, to the constraints

$$-\frac{N_C}{8\pi^2}+8\sqrt{2}f_Vh_V-4f_V^2\sigma_V=0, \\ -\frac{N_C}{24\pi^2}+8\sqrt{2}f_Ah_A-4f_A^2\sigma_A=0. \qquad (7.5)$$

However, at order $1/\lambda^4$, we again observe that $\mathcal{H}_V^{\text{res}}$ and $\mathcal{H}_A^{\text{res}}$ are not consistent with the OPE result, since the terms $\sim 1/(p^2q^2)$ are missing in (7.4).

7.2 One large momentum

In Sect. 3 we have also derived the short-distance properties of the Green's functions $\langle VAP\rangle, \langle VVP\rangle$ and $\langle AAP\rangle$ when the relative distance between only two of the currents becomes small. In certain physical applications only this limit is relevant and we shall now discuss the corresponding constraints on the parameters in the resonance Lagrangian. We note, however, that it is not possible to satisfy simultaneously *all* the constraints given below. Moreover, some of these constraints are in contradiction with the relations derived in the previous section. These inconsistencies can again be traced back to the fact that

[6] If a tensor-field formulation is used for the vector and axial-vector resonances, the contributions from the resonances start already at order p^4 and lead directly to the usual estimates for the L_i. On the other hand, several couplings which appear in the resonance Lagrangian of [38], like α_V, β_V or α_A, cannot be written down in the tensor-field representation, at least not without introducing additional derivatives

the Green's functions derived from the resonance Lagrangian do not correctly reproduce the QCD short-distance behavior.

The constraints on the functions $\mathcal{F}$ and $\mathcal{G}$ from (3.13) and (3.14), when the space-time arguments of the vector and axial-vector currents coincide, can be satisfied provided the resonance parameters obey the relations

$$A^{(2)} = \sqrt{2}\frac{\alpha_V}{f_A}, \quad B = -\sqrt{2}\frac{\beta_V}{f_A}. \tag{7.6}$$

Furthermore, we recover the usual resonance estimate $L_{10} = -f_V^2/4 + f_A^2/4$ and the first Weinberg sum rule $M_V^2 f_V^2 = F_0^2 + M_A^2 f_A^2$. Imposing the relation $L_9 = f_V g_V/2$, we furthermore get the relations

$$A^{(3)} = \sqrt{2}\frac{\alpha_A}{f_V}, \tag{7.7}$$
$$M_V^2 f_V^2 - M_V^2 f_V g_V + \sqrt{2}M_A^2 f_V \alpha_V + \sqrt{2}M_V^2 f_A \alpha_A = 0.$$

The second limit, when the distance between the vector current and the pseudoscalar density becomes small, corresponding to the constraints (3.15), can be satisfied provided

$$A^{(3)} = \sqrt{2}\frac{\alpha_A}{f_V}, \quad \beta_V = 0, \quad B = 0. \tag{7.8}$$

We also get the result $L_9 = f_V g_V/2$. Imposing the usual resonance estimate for L_{10}, we obtain the additional relations

$$\sqrt{2}f_V\alpha_V = -\frac{f_A^2}{2}, \quad A^{(2)} = \sqrt{2}\frac{\alpha_V}{f_A}. \tag{7.9}$$

Finally, the constraints (3.16), when the arguments of the axial-vector current and the pseudoscalar density coincide, lead again to the relations (7.6). Using in addition the usual resonance estimates for L_9 and L_{10}, we obtain

$$A^{(3)} = \sqrt{2}\frac{\alpha_A}{f_V}, \quad f_V g_V - \frac{f_V^2}{2} - \sqrt{2}f_A\alpha_A = \frac{F_0^2}{2M_V^2}. \tag{7.10}$$

We note that we recover in the first case, corresponding to (3.13) and (3.14), all the constraints from the leading terms in the expansion in $1/\lambda$, when all momenta in $\mathcal{F}$ and $\mathcal{G}$ become large. In the latter two cases we get, however, only a subset of these relations.

Equations (7.6)–(7.10) arising from the three different limits are perfectly compatible. If however we combine them with the first relation in (7.3) from the OPE constraint when all momenta in $\mathcal{G}$ become large, we find a contradiction.

We now turn to the functions $\langle VVP\rangle$ and $\langle AAP\rangle$. We first consider the case when the two vector currents are taken at the same point, which is relevant, for instance, in the decay $P \to l^+l^-$, where $P = \pi^0, \eta$ (see the discussion in [43]) or when the space-time arguments of the two axial-vector currents in $\langle AAP\rangle$ coincide. The corresponding constraint (3.17) can be satisfied, provided the resonance parameters fulfill the relations

$$\sqrt{2}f_X h_X = c_X\frac{N_C}{32\pi^2} - \frac{1}{8}\frac{F_0^2}{M_X^2},$$
$$f_X^2\sigma_X = c_X\frac{N_C}{32\pi^2} - \frac{1}{4}\frac{F_0^2}{M_X^2}, \quad X = V, A, \tag{7.11}$$

with $c_V = 1, c_A = 1/3$. In the case when the space-time argument of one of the vector currents and of the pseudoscalar density in $\langle VVP\rangle$ coincide, the constraint from (3.18) can be satisfied, if

$$\sqrt{2}f_V h_V = \frac{N_C}{32\pi^2} - \frac{1}{4}\frac{F_0^2}{M_V^2}, \quad f_V^2\sigma_V = \frac{N_C}{32\pi^2} - \frac{1}{2}\frac{F_0^2}{M_V^2}, \tag{7.12}$$

where we have also used the short-distance constraint (3.5) on $\Pi_{VT}(p^2)$. The same limit in $\langle AAP\rangle$, see (3.19), yields the relations

$$\sqrt{2}f_A h_A = \frac{N_C}{96\pi^2}, \quad f_A^2\sigma_A = \frac{N_C}{96\pi^2}. \tag{7.13}$$

We recover in all cases the constraints (7.5) when all momenta in $\mathcal{H}_{V,A}$ become large. We note, however, that although one can individually satisfy the constraints from the OPE, the relations (7.12) and (7.13) are incompatible with (7.11).

8 Resonance contributions to pion form factors

In this section, we discuss a few phenomenological applications of the various ansätze considered previously. The first two examples, the vector form factor of the pion and the radiative pion decay, involve the low-energy constants C_i that were determined in Sect. 5. The third example, the pion–photon–photon transition form factor, illustrates a situation where the MHA goes beyond the simplest LMD approximation.

8.1 Vector form factor of the pion

There are two combinations of renormalized low-energy constants c_i^r from the chiral Lagrangian $\mathcal{L}_6$ that enter in the vector form factor of the pion; see [50],

$$r_{V1}^r = -16c_6^r - 4c_{35}^r - 8c_{53}^r,$$
$$r_{V2}^r = -4c_{51}^r + 4c_{53}^r.$$

The resonance estimates for r_{V1}^r and r_{V2}^r given in [51] read (recall that $f_\chi \equiv \beta_V$)

$$r_{V1}^{r,\text{res}}(\mu = M_V) = \frac{2\sqrt{2}f_\chi f_V F^2}{M_V^2},$$
$$r_{V2}^{r,\text{res}}(\mu = M_V) = \frac{g_V f_V F^2}{M_V^2}. \tag{8.1}$$

Using the resonance estimates for the constants c_i^{res} given in (6.4) we obtain the same result for $r_{V2}^{r,\text{res}}$, if we identify the pion decay constant F with F_0.

With the LMD estimates from (5.3) we get

$$r_{V2}^{r,\mathrm{LMD}}(\mu = M_V) = \frac{F_0^4}{M_V^4}, \qquad (8.2)$$

which agrees with the result given in (8.1), if we use the relation $F_V G_V = F_0^2$ which is valid within the LMD ansatz. From the present analysis, we cannot obtain an estimate for r_{V1}^r, which describes the quark mass corrections to the value of the vector form factor at vanishing momentum transfer, see [51].

8.2 The decay $\pi \to e\nu_e\gamma$

The decay $\pi(p) \to e\nu_e\gamma(q)$ is described by two form factors V and A. The contribution from $\mathcal{L}_6$ to A can be written as [52,50]

$$A((p-q)^2) = M_\pi^2 r_{A1}^r + (p\cdot q) r_{A2}^r,$$

with

$$r_{A1}^r = 48c_6^r - 16c_{34}^r + 8c_{35}^r - 8c_{44}^r + 16c_{46}^r - 16c_{47}^r + 8c_{50}^r,$$
$$r_{A2}^r = 8c_{44}^r - 16c_{50}^r + 4c_{51}^r.$$

With the estimates for the low-energy constants c_i^r obtained from the resonance Lagrangian, (6.4), we obtain

$$r_{A2}^{r,\mathrm{res}}(\mu = M_V) = F^2 \frac{2}{M_A^2}\left(f_A^2 + f_A\alpha_A 2\sqrt{2}\right) \approx 0.55\cdot 10^{-4}, \qquad (8.3)$$

where we used the same values $f_A = 0.080, \alpha_A = -6.66\cdot 10^{-3}$, as in [52] and $F \approx F_0 = 92.4\,\mathrm{MeV}$, and $M_A = 1230\,\mathrm{MeV}$.

On the other hand, using the LMD estimates given in Eqs. (5.3) we obtain

$$r_{A2}^{r,\mathrm{LMD}}(\mu = M_V) = \frac{F_0^4}{M_V^2 M_A^2} - 2\frac{F_0^4}{M_A^4} \approx 0.18\cdot 10^{-4}. \qquad (8.4)$$

The LMD estimate differs by a factor three from the value given in (8.3). In fact, since we obtain a different relative sign in our result for $r_{A2}^{r,\mathrm{res}}$ as compared to the expression given in [52], the LMD estimate is even a factor of five smaller than the value $r_{A2}^{r,\mathrm{res}}(\mu = M_V) \approx 0.89\cdot 10^{-4}$ used in that paper.

Since the same combination of resonance parameters that determines r_{A2}^r also enters in the decay amplitude for $a_1 \to \pi\gamma$, the LMD ansatz predicts a decay rate more than an order of magnitude smaller than the usual value [53]. However, the experimental situation concerning this decay is far from being clear [54], see the remarks following (62) in [33].

8.3 Pion–photon–photon transition form factor

Up to now, we have shown that a very minimal ansatz allows to take into account the *leading* asymptotic behaviors of the three-point functions $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$. This simple representation would in general certainly not be sufficient if additional information were added. However, within the large-N_C framework considered here, one always has the freedom to go beyond the LMD representation and to add other zero-width resonance states. In this section, we wish to illustrate this point by considering the form factor $\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)$ that describes the transition between a pion and two (possibly off-shell) photons in the chiral limit. This form factor is defined as

$$\mathrm{i}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(p)\rangle = \epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta \mathcal{F}_{\pi\gamma^*\gamma^*}(q^2,(p-q)^2) \qquad (8.5)$$

or still

$$\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\lim_{(q_1+q_2)^2\to 0}(q_1+q_2)^2\mathcal{H}_V(q_1^2,q_2^2,(q_1+q_2)^2).$$

For both photons on-shell, the value of this form factor is fixed by the Wess–Zumino–Witten anomaly term,

$$\mathcal{F}_{\pi\gamma^*\gamma^*}(0,0) = -\frac{N_C}{12\pi^2 F_0}.$$

Many studies have been devoted to this form factor in the past, see [42,55] and references therein. In particular, its behavior in the limit $Q^2 \to \infty$, ω fixed, with $-Q^2 = (q_1^2 + q_2^2)$, $\omega = (q_1^2 - q_2^2)/(q_1^2 + q_2^2)$, has been investigated, with the result

$$\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2,q_2^2) = -\frac{4F_0}{3}\frac{f(\omega)}{Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right), \qquad (8.6)$$

for $-1 < \omega < 1$. The function $f(\omega)$ can be expressed in terms of the pion distribution function $\varphi_\pi(u)$ [42]

$$f(\omega) = \int_0^1 \mathrm{d}u \frac{\varphi_\pi(u)}{(1-u)(1+\omega)+u(1-\omega)}\left[1 + \mathcal{O}(\alpha_s)\right],$$

normalized as

$$\int_0^1 \mathrm{d}u\varphi_\pi(u) = 1.$$

This last condition is sufficient in order to study the limit $-q_1^2 = -q_2^2 = Q^2/2 \to \infty$ and leads to the result [56]

$$\mathcal{F}_{\pi\gamma^*\gamma^*}\left(-\frac{Q^2}{2}, -\frac{Q^2}{2}\right) = -\frac{4F_0}{3}\frac{1}{Q^2} + \mathcal{O}\left(\frac{1}{Q^4}\right).$$

The function $\varphi_\pi(u)$ is only known asymptotically, and this asymptotic expression is reliable only for ω not too large [57], e.g. $|\omega| < 1/2$. The case of one on-shell photon corresponds to $|\omega| = 1$, so that the coefficient of the $1/Q^2$ fall-off of the form factor $\mathcal{F}_{\pi\gamma^*\gamma^*}(-Q^2, 0)$ is actually not known. Depending on the assumptions made or the ansätze considered for $\varphi_\pi(u)$, different results have been obtained in the literature.

The LMD ansatz for $\mathcal{H}_V(q_1^2, q_2^2, (q_1+q_2)^2)$ reproduces these results for $|\omega| < 1$ with $f^{\mathrm{LMD}}(\omega) = 1/(1-\omega^2)$. On the other hand, taking $q_2^2 = 0$ and letting $Q^2 = -q_1^2$ become large, we obtain

$$\mathcal{F}^{\mathrm{LMD}}_{\pi\gamma^*\gamma^*}(-Q^2, 0) \sim \text{const.}$$

In order to recover the $1/Q^2$ behavior for $|\omega| = 1$, we need to go beyond the LMD approximation and add a second vector resonance (adding a pseudoscalar resonance [32] would not help to improve the situation in the present case, since the $1/Q^2$ behavior forces $\hat{h}_1 = 0$ in the ansatz for $\mathcal{H}_V$, in contradiction with the relation (4.16)). From (4.9) we obtain

$$\begin{aligned}\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{F_0}{3}\Big\{ &\Big(q_1^2 q_2^2 \left[q_1^2 + q_2^2\right] + h_1 (q_1^2+q_2^2)^2 \\ &+ h_2 q_1^2 q_2^2 + h_5(q_1^2+q_2^2) + h_7 \Big) \Big/ \Big((q_1^2 - M_{V_1}^2) \\ &\times (q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2) \Big) \Big\}. \qquad (8.7)\end{aligned}$$

The behavior of $\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)$ for Q^2 large and ω fixed, with $|\omega| < 1$, is the same as in the case of $\mathcal{F}^{\mathrm{LMD}}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)$. However, if we now set $q_2^2 = 0$, we obtain

$$\begin{aligned}&\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}(-Q^2, 0) \\ &= \frac{F_0}{3} \frac{1}{M_{V_1}^2 M_{V_2}^2} \frac{h_1 Q^4 - h_5 Q^2 + h_7}{(Q^2 + M_{V_1}^2)(Q^2 + M_{V_2}^2)}. \qquad (8.8)\end{aligned}$$

Imposing that this expression exhibits the $1/Q^2$ behavior for large Q^2 requires that h_1 vanishes, which then gives

$$\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}(-Q^2, 0) = -\frac{2F_0}{3} \frac{1}{Q^2} \frac{h_5}{2M_{V_1}^2 M_{V_2}^2} + \mathcal{O}\left(\frac{1}{Q^4}\right). \qquad (8.9)$$

In the absence of a reliable prediction for the coefficient that governs the $1/Q^2$ behavior, this still leaves the parameter h_5 undetermined. Additional information may be obtained from the fact that the form factor $\mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)$ can also be related to the decay $\rho^+ \to \pi^+\gamma$, whose amplitude is given by

$$\mathcal{A}(\rho^+ \to \pi^+\gamma) = \frac{e}{3} \lim_{q_1^2 \to M_V^2} \lim_{q_2^2 \to 0} \frac{(q_1^2 - M_V^2)}{F_V M_V} \mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2). \qquad (8.10)$$

Note that we obtain the same relation for $\mathcal{A}(\rho^0 \to \pi^0\gamma)$. In the latter case, however, $\rho-\omega$ mixing would have to be taken into account for a realistic calculation. Furthermore, we have defined the coupling F_V of the ρ meson to the vector current by

$$\langle 0 | V_\mu^a(0) | \rho^b(p) \rangle = \delta^{ab} F_V M_V \varepsilon_\mu,$$

where ε_μ denotes the polarization vector of the ρ meson. From (8.10) we obtain (see also [32])

$$\begin{aligned}&-\left(\frac{2eF_V}{M_V}\right) \frac{\mathcal{A}(\rho^+ \to \pi^+\gamma)}{\mathcal{A}(\pi^0 \to \gamma\gamma)} \qquad (8.11) \\ &= \frac{\lim_{q_1^2 \to M_V^2, q_2^2 \to 0} (q_1^2 - M_V^2) \mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)}{\lim_{q_1^2 \to 0, q_2^2 \to 0} (q_1^2 - M_V^2) \mathcal{F}_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)} = 1 + x.\end{aligned}$$

The observed value $\Gamma = 68 \pm 7\,\mathrm{keV}$ for the decay width $\rho^+ \to \pi^+\gamma$ [53] yields $x = 0.022 \pm 0.051$ [32]. The LMD ansatz for $\mathcal{F}_{\pi\gamma^*\gamma^*}$ leads to $x_{\mathrm{LMD}} = -(4\pi^2/N_C)(F_0^2/M_V^2) = -0.19$, far from the experimental value [58]. Starting instead from the form factor $\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}$ leads to

$$\begin{aligned}(1+x)_{\mathrm{LMD}+V} &= \frac{1}{(1 - M_{V_1}^2/M_{V_2}^2)} \\ &\times \left(1 - \frac{4\pi^2}{N_C} \frac{F_0^2}{M_{V_1}^2} \left[\frac{M_{V_1}^2}{M_{V_2}^2} \frac{h_1}{M_{V_2}^2} + \frac{h_5}{M_{V_2}^4}\right]\right).\end{aligned}$$

Setting $h_1 = 0$ and solving for h_5 gives $h_5 = 6.3 \pm 0.9\,\mathrm{GeV}^4$, where we have taken $M_{V_1} = 769\,\mathrm{MeV}$, $M_{V_2} = 1465\,\mathrm{MeV}$ for the resonance masses and $F_0 = 92.4\,\mathrm{MeV}$.

Actually, the form factor $\mathcal{F}_{\pi\gamma^*\gamma^*}(-Q^2, 0)$ has been measured in the space-like region by the CLEO collaboration, see Table 1 in [59], over a wide range of Q^2, $1.5\,\mathrm{GeV}^2 \leq Q^2 \leq 9\,\mathrm{GeV}^2$. A fit of the expression $\mathcal{F}^{\mathrm{LMD}+V}_{\pi\gamma^*\gamma^*}(-Q^2, 0)$ with $h_1 = 0$ to these data yields

$$h_5 = 6.93 \pm 0.26\,\mathrm{GeV}^4, \qquad (8.12)$$

with $\chi^2/\mathrm{dof} = 7.00/14 = 0.50$. Keeping also h_1 as a free parameter yields instead

$$\begin{aligned}h_1 &= -0.01 \pm 0.16\,\mathrm{GeV}^2, \\ h_5 &= 6.88 \pm 0.61\,\mathrm{GeV}^4,\end{aligned}$$

with $\chi^2/\mathrm{dof} = 6.99/13 = 0.54$. The results for h_5 from both fits are compatible with the value extracted above from the decay $\rho^+ \to \pi^+\gamma$.

Finally, one might ask to which extent the inclusion of a second vector resonance into the ansatz for $\mathcal{H}_V$ modifies the determination of the corresponding combinations of low-energy constants. The only combination which can be fixed from the knowledge of $\mathcal{F}_{\pi\gamma^*\gamma^*}(-Q^2, 0)$ alone is A_{V,p^2} (see (6.6)),

$$A^{\mathrm{LMD}+V}_{V,p^2} = \frac{1}{8} \frac{F_0^2}{M_{V_1}^4} \frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2} \frac{1}{M_{V_1}^2} \left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right), \qquad (8.13)$$

since the other combination, $A^{\mathrm{LMD}+V}_{V,(p+q)^2}$, involves h_6. With the value of h_5 obtained in (8.12), we find, in units of $10^{-4}/F_0^2$, $A^{\mathrm{LMD}+V}_{V,p^2} = -1.36$, i.e. about 20% away from our LMD estimate reported in Table 3. The difference is well within the 30% relative error that we attribute to the approximations considered there.

Another approach was followed in [32], where an additional pseudoscalar resonance π' was included in the ansatz for $\mathcal{H}_V$ that satisfies the OPE constraints (LMD+P, see (4.15)). As noted above, this ansatz will, however, not correctly reproduce the $1/Q^2$ behavior of $\mathcal{F}_{\pi\gamma^*\gamma^*}(-Q^2, 0)$ at large Q^2. In this case, $A_{V,p^2} = -N_C(1+x)/32\pi^2 M_V^2$ and from (8.11) one obtains the result, again in units of $10^{-4}/F_0^2$, $A^{\mathrm{LMD}+P}_{V,p^2} = -1.40$, close to our LMD+V estimate from the fit to the CLEO data. In a similar way, the low-energy constant $A_{V,(p+q)^2}$ receives within the LMD+P ansatz an additional contribution proportional to the decay amplitude $\mathcal{A}(\pi' \to \gamma\gamma)$ [32]. Since this decay has not yet been observed experimentally, our LMD estimate for $A_{V,(p+q)^2}$ from Table 3 is probably not strongly modified by the addition of a pseudoscalar resonance.

The low-energy constant A_{V,p^2} also describes the contributions from the counterterms at order p^6 to the slope b_π of the form factor at the origin

$$b_\pi \equiv \left(\frac{1}{\mathcal{A}(\pi^0 \to \gamma\gamma^*(q^2))} \frac{\mathrm{d}}{\mathrm{d}q^2} \mathcal{A}(\pi^0 \to \gamma\gamma^*(q^2)) \right)_{q^2=0} .$$

At $\mathcal{O}(p^6)$ this slope also receives contributions from chiral loops, therefore, $b_\pi = b_\pi^{\mathrm{loops}} + b_\pi^{CT}$, with $b_\pi^{CT} = -32\pi^2 A_{V,p^2}/N_C$. The Particle Data Group gives the value $a_\pi \equiv M_{\pi^0}^2 b_\pi = 0.032 \pm 0.004$ [53]. From our LMD+V estimate above we obtain $b_\pi^{CT,\mathrm{LMD}+V} = 1.67\,\mathrm{GeV}^{-2}$ or, equivalently, $a_\pi^{CT,\mathrm{LMD}+V} = 0.031$. To this value, one should add the contribution from the chiral logarithms, evaluated at the scale $\mu \sim M_\rho$, $a_\pi^{\mathrm{loops}} \sim 0.005$ [60], which represents a 20% effect. We note that the value for a_π used by the PDG is essentially the one reported by the CELLO collaboration [61]. In the latter paper a simple VMD-inspired pole ansatz was fitted with their data for the form factor in the space-like region for $0.5\,\mathrm{GeV}^2 \leq Q^2 \leq 2.7\,\mathrm{GeV}^2$, not taking into account contributions from Goldstone boson loops at low Q^2.

9 Conclusions

In this article we have studied the QCD three-point functions $\langle VAP \rangle, \langle VVP \rangle$ and $\langle AAP \rangle$ in the three flavor chiral limit in order to obtain resonance estimates for some of the low-energy constants that appear at $\mathcal{O}(p^6)$ in chiral perturbation theory in the meson sector (even and odd intrinsic parity). We have compared the results that have been obtained in the literature using a Lagrangian that includes resonance fields [38] with those evaluated within the framework of an approximation of large-N_C QCD combined with information on short-distance properties. In certain cases, we have found substantially different results for the estimates of the low-energy constants obtained with these two methods. We have pointed out that this is due to the fact that the Green's functions derived from the resonance Lagrangian do not correctly reproduce the QCD short-distance behavior. This defect can be repaired, but at the expense of introducing, into the resonance Lagrangian, certain local contributions. The difference with the similar situation at the $\mathcal{O}(p^4)$ level lies in the fact that these local contributions *cannot be restricted to terms already present* in the $\mathcal{O}(p^6)$ chiral Lagrangian, but also involve terms with resonance fields and higher order derivatives. This feature, already noticed in a particular case in [31], seems to be of a general character. A general construction remains to be done, and appears to be a much more complicated task than at $\mathcal{O}(p^4)$.

We note that although in general the short-distance behavior of the Green's functions derived from the resonance Lagrangian [38] is incompatible with QCD, one can sometimes reproduce the results from the operator product expansion, by adjusting the resonance parameters accordingly. There are, however, certain cases where this is not possible. The numerical values for the low-energy constants can then be very different. In particular, whereas both methods lead to identical estimates for the resonance contributions at order p^6 in the vector form factor of the pion $F_V^\pi(q^2)$, our estimate for the resonance contribution in one of the form factors for the decay $\pi \to e\nu_e\gamma$ is a factor of five smaller than the results quoted in [52].

Of course one might argue that the short-distance behavior of Green's functions, i.e. their behavior at very high energies, is irrelevant for the determination of low-energy constants in chiral Lagrangians starting from a resonance Lagrangian that is supposed to be valid only in the intermediate energy region anyway. We think, however, that taking into account the QCD short-distance constraints is a good guiding principle to avoid (some of) the ambiguities when working with resonance fields, as was shown at order p^4 in [20]. Moreover, in certain cases one needs also integrals of these Green's functions, for instance, to estimate the low-energy constants that appear if virtual photons [34] or leptons [37] are included in chiral perturbation theory, see the discussion in [33,36]. The case of the counterterms of the effective Lagrangians [62] in the $|\Delta S| = 1$ or $|\Delta S| = 2$ sectors of the standard model presents very similar features [63,64]. In these applications it is crucial that the Green's functions respect the short-distance constraints in order to obtain ultraviolet finite results or to implement a correct matching between long and short distances.

Finally, we wish to add a few remarks concerning the methodology followed in the present study. We shall not come back on the use of the large-N_C framework, which seems to be unavoidable once correlators of higher rank than two-point functions need to be considered. Our approach deviates however from a full large-N_C limit of QCD by at least two aspects. For one thing, we have only imposed constraints coming from the leading (for the three-point functions) and next-to-leading (for the vertex functions) short-distance properties of QCD. We have considered neither the effects of higher dimension operators in the Wilson expansions, nor have we included QCD corrections to the short-distance behavior (recall however, that for the cases treated here, the corresponding Wilson coefficients had no anomalous dimensions). For the other thing, we have truncated the mesonic spectrum of large-N_C QCD to the minimal number of resonances necessary in order to fulfill the short-distance constraints that were considered. Both approximations are to a large extent interdependent. It is clear that, say, the simplest LMD ansatz will at some point fail to reproduce the subleading short-distance behavior. On the other hand, the knowledge of the subdominant operators in the OPE, or other constraints, will fix additional parameters that have to be introduced if one goes beyond the LMD approximation. In this sense, the framework within which we have been working is, given the necessary amount of work, improvable. As an illustration, we have considered the pion–photon–photon form factor $F_{\pi\gamma^*\gamma^*}(-Q^2,0)$. The experimentally observed [59] $1/Q^2$ fall-off of the form factor cannot be reproduced with our LMD ansatz for the invariant function $\mathcal{H}_V$. Including one additional vector resonance, we obtain a representation for this form factor that fulfills all constraints from the leading terms in the operator product expansion and that fits the experimental results successfully if we adjust some of the unknown parameters that enter in the generalized ansatz for $\mathcal{H}_V$. We note that this is not possible if one includes a pseudoscalar resonance instead. This new ansatz, with the phenomenologically determined parameters, is furthermore compatible with the observed decay rate for $\rho^+ \to \pi^+\gamma$. Finally, the estimate for one combination of low-energy constants A_i in the odd-intrinsic parity sector changes by about 20% with this new ansatz. This difference is within the 30% relative error that we attribute to the approximations considered here. The determination of the low-energy constants seems thus to be stable against the inclusion of higher mass resonances.

It is clear that the three-point functions we have considered do not, by far, exhaust the whole set of low-energy constants of the chiral Lagrangian at the $\mathcal{O}(p^6)$ level. Other three-point functions, as well as higher correlators, need to be studied in a similar way for that purpose. Some of them (in particular, those describing the low-energy constants related to quark mass corrections) will include the scalar densities. Recent studies have emphasized that in the 0^{++} channel, predictions relying on the large-N_C picture might not be very reliable [65,64], because of the strong $\pi\pi$ interaction in the S-wave. It has been suggested that a more appropriate treatment would rather require to consider the limit where both N_C and N_F, the number of light flavors, become large, with a fixed ratio N_F/N_C [66]. Also, in the case of two- and three-point functions, the quantum numbers of the resonances that can contribute in the large-N_C limit are entirely fixed by the quantum numbers of the quark bilinears involved. This is no longer true for n-point functions with $n \geq 4$ [21]. We leave these and other interesting issues for future work.

Acknowledgements. We thank J. Gasser, H. Leutwyler, B. Moussallam, M. Perrottet and J. Stern for useful discussions. We are grateful to S. Peris, J. Prades and E. de Rafael for discussions and comments on the manuscript. This work has been supported in part by Schweizerischer Nationalfonds and by TMR, EC-Contract No. ERBFMRX-CT980169 (EURODAΦNE).

Appendix

A Operator product expansion for vertex functions

In this appendix we discuss the short-distance behavior of the vertex functions

$$(\Gamma_{VA})^{abc}_{\mu\nu}(q,p) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{V^a_\mu(x)A^b_\nu(0)\}|\pi^c(p)\rangle,$$

$$(\Gamma_{VP})^{abc}_{\mu}(q,p) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{V^a_\mu(x)P^b(0)\}|\pi^c(p)\rangle,$$
$$(\Gamma_{VV})^{abc}_{\mu\nu}(q,p) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{V^a_\mu(x)V^b_\nu(0)\}|\pi^c(p)\rangle,$$
$$(\Gamma_{AA})^{abc}_{\mu\nu}(q,p) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{A^a_\mu(x)A^b_\nu(0)\}|\pi^c(p)\rangle. \quad \text{(A.1)}$$

They are related, according to the LSZ reduction formula, to the $\langle VAP\rangle$, $\langle VVP\rangle$ and $\langle AAP\rangle$ three-point functions as follows:

$$(\Gamma_{VA})^{abc}_{\mu\nu}(q,p) = \mathrm{i}\frac{F_0}{\langle\bar\psi\psi\rangle_0}f^{abc} \times\Bigg[\langle\bar\psi\psi\rangle_0\left(\frac{(2p-q)_\mu(p-q)_\nu}{(p-q)^2} - \eta_{\mu\nu}\right) + P_{\mu\nu}(q,p-q)\tilde{\mathcal{F}}(q^2,q\cdot p) + Q_{\mu\nu}(q,p-q)\tilde{\mathcal{G}}(q^2,q\cdot p)\Bigg],$$
$$\tilde{\mathcal{F}}(q^2,q\cdot p) = \lim_{p^2\to 0} p^2\mathcal{F}(q^2,(p-q)^2,p^2),$$
$$\tilde{\mathcal{G}}(q^2,q\cdot p) = \lim_{p^2\to 0} p^2\mathcal{G}(q^2,(p-q)^2,p^2), \quad \text{(A.2)}$$

and

$$(\Gamma_{VP})^{abc}_{\mu}(q,p) = \frac{-1}{F_0}f^{abc} \times\left[-\langle\bar\psi\psi\rangle_0\frac{(q-2p)_\mu}{(q-p)^2} + (q^2p_\mu - (q\cdot p)q_\mu)\check{\mathcal{G}}(q^2,q\cdot p)\right],$$
$$\check{\mathcal{G}}(q^2,q\cdot p) = \lim_{p^2\to 0} p^2\mathcal{G}(q^2,p^2,(q-p)^2), \quad \text{(A.3)}$$

since the invariant function $\mathcal{F}(q^2,p^2,(q-p)^2)$ does not contain a $1/p^2$ Goldstone boson pole. Finally,

$$(\Gamma_{VV})^{abc}_{\mu\nu}(q,p) = \mathrm{i}\frac{F_0}{\langle\bar\psi\psi\rangle_0}\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta d^{abc}\tilde{\mathcal{H}}_V(q^2,q\cdot p),$$
$$(\Gamma_{AA})^{abc}_{\mu\nu}(q,p) = \mathrm{i}\frac{F_0}{\langle\bar\psi\psi\rangle_0}\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta d^{abc}\tilde{\mathcal{H}}_A(q^2,q\cdot p),$$
$$\tilde{\mathcal{H}}_X(q^2,q\cdot p) = \lim_{p^2\to 0} p^2\mathcal{H}_X(q^2,(p-q)^2,p^2), \quad X = V, A, \quad \text{(A.4)}$$

with $\tilde{\mathcal{H}}_X(q^2,q\cdot p) = \tilde{\mathcal{H}}_X(q^2-2q\cdot p,-q\cdot p)$, as a consequence of (2.4).

For general Dirac matrices $\Gamma_{1,2}$ we obtain the OPE

$$\lim_{\lambda\to\infty}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}(\lambda q)\cdot x} \times\langle 0|T\left\{\left(\bar\psi\Gamma_1\frac{\lambda^a}{2}\psi\right)(x)\left(\bar\psi\Gamma_2\frac{\lambda^b}{2}\psi\right)(0)\right\}|\pi^c(p)\rangle$$
$$= \lim_{\lambda\to\infty}\int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}(\lambda q)\cdot x}(\Gamma_1)_\alpha{}^\beta\left(\frac{\lambda^a}{2}\right)_{IJ}(\Gamma_2)_\gamma{}^\delta\left(\frac{\lambda^b}{2}\right)_{KL} \times\Big[\mathrm{i}S(x)_\beta{}^\gamma\delta_{JK}\langle 0|:\bar\psi^\alpha_I(x)\psi_{\delta,L}(0):|\pi^c(p)\rangle + \mathrm{i}S(-x)_\delta{}^\alpha\delta_{IL}\langle 0|:\bar\psi^\gamma_K(0)\psi_{\beta,J}(x):|\pi^c(p)\rangle\Big]$$

$$
\begin{aligned}
&+\cdots \\
&= \frac{\mathrm{i}}{\lambda}(\Gamma_1)_\alpha{}^\beta \left(\frac{\lambda^a}{2}\right)_{IJ} (\Gamma_2)_\gamma^\delta \left(\frac{\lambda^b}{2}\right)_{KL} \\
&\times \left[\left(\frac{\not{q}}{q^2}\right)_\beta^{\ \gamma} \delta_{JK}\langle 0| : \bar{\psi}_I^\alpha(0)\psi_{\delta,L}(0) : |\pi^c(p)\rangle \right.\\
&\left. - \left(\frac{\not{q}}{q^2}\right)_\delta^{\ \alpha} \delta_{IL}\langle 0| : \bar{\psi}_K^\gamma(0)\psi_{\beta,J}(0) : |\pi^c(p)\rangle\right] \\
&+\frac{1}{\lambda^2}(\Gamma_1)_\alpha{}^\beta \left(\frac{\lambda^a}{2}\right)_{IJ} (\Gamma_2)_\gamma{}^\delta \left(\frac{\lambda^b}{2}\right)_{KL} \\
&\times \frac{\partial}{\partial q^\rho}\left[\left(\frac{\not{q}}{q^2}\right)_\beta^{\ \gamma} \delta_{JK}\langle 0| : (D_\rho\bar{\psi})_I^\alpha(0)\psi_{\delta,L}(0) : |\pi^c(p)\rangle \right.\\
&\left. - \left(\frac{\not{q}}{q^2}\right)_\delta^{\ \alpha} \delta_{IL}\langle 0| : \bar{\psi}_K^\gamma(0)(D_\rho\psi)_{\beta,J}(0) : |\pi^c(p)\rangle\right] \\
&+\mathcal{O}\left(\frac{1}{\lambda^3}\right), \qquad (\mathrm{A}.5)
\end{aligned}
$$

up to possible $\mathcal{O}(\alpha_s)$ corrections. Note that the indices $I, J, \ldots$ label both flavor and color and that in this Appendix $(\lambda^a/2)_{IJ}$ denote the Gell-Mann matrices in flavor space and the unit matrix in color space, with $\mathrm{tr}(\lambda^a\lambda^b) = 2N_C\delta^{ab}$. From invariance under parity, Lorentz, flavor $SU(3)_V$ and color $SU(3)_C$ transformations the matrix elements involved in (A.5) can be expressed as

$$
\begin{aligned}
&\langle 0| : \bar{\psi}_I^\alpha(0)\psi_{\delta,L}(0) : |\pi^c(p)\rangle \\
&= \sum_d \left(\frac{\lambda^d}{2}\right)_{LI} \\
&\times \left[(\mathrm{i}\gamma_5)_\delta{}^\alpha \langle 0| : \left(\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right)\right.\\
&\left.+(\gamma_\sigma\gamma_5)_\delta{}^\alpha \langle 0| : \left(\bar{\psi}\gamma^\sigma\gamma_5\frac{\lambda^d}{2}\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0| : (D_\rho\bar{\psi})_I^\alpha(0)\psi_{\delta,L}(0) : |\pi^c(p)\rangle \\
&= \sum_d \left(\frac{\lambda^d}{2}\right)_{LI} \\
&\times \left[(\mathrm{i}\gamma_5)_\delta{}^\alpha \langle 0| : \left(D_\rho\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right)\right.\\
&+(\gamma_\sigma\gamma_5)_\delta{}^\alpha \langle 0| : \left(D_\rho\bar{\psi}\gamma^\sigma\gamma_5\frac{\lambda^d}{2}\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right) \\
&\left.+(\gamma_5\sigma_{\rho\sigma})_\delta{}^\alpha \langle 0| : \left(D^\sigma\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{6N_C}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0| : \bar{\psi}_K^\gamma(0)(D_\rho\psi)_{\beta,J}(0) : |\pi^c(p)\rangle \\
&= \sum_d \left(\frac{\lambda^d}{2}\right)_{JK} \\
&\left[(\mathrm{i}\gamma_5)_\beta{}^\gamma \langle 0| : \left(\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}D_\rho\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right)\right.\\
&+(\gamma_\sigma\gamma_5)_\beta{}^\gamma \langle 0| : \left(\bar{\psi}\gamma^\sigma\gamma_5\frac{\lambda^d}{2}D_\rho\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{-1}{2N_C}\right) \\
&\left.+(\gamma_5\sigma_{\rho\sigma})_\beta{}^\gamma \langle 0| : \left(\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}D^\sigma\psi\right)(0) : |\pi^c(p)\rangle \left(\frac{1}{6N_C}\right)\right].
\end{aligned}
$$

In the above, we have also made use of the equations of motion in the chiral limit, $\not{D}\psi = 0$.

Invariance under space-time translations, parity and charge conjugation further yields

$$
\begin{aligned}
&\langle 0| \left(D_\rho\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}\psi\right)(0)|\pi^c(p)\rangle \\
&\quad = \langle 0| \left(\bar{\psi}\mathrm{i}\gamma_5\frac{\lambda^d}{2}D_\rho\psi\right)(0)|\pi^c(p)\rangle = \frac{\mathrm{i}}{2}p_\rho\frac{\langle\bar{\psi}\psi\rangle_0}{F_0}\delta^{dc},
\end{aligned}
$$

$$
\begin{aligned}
&\langle 0| \left(D_\rho\bar{\psi}\gamma_\sigma\gamma_5\frac{\lambda^d}{2}\psi\right)(0)|\pi^c(p)\rangle \\
&\quad = \langle 0| \left(\bar{\psi}\gamma_\sigma\gamma_5\frac{\lambda^d}{2}D_\rho\psi\right)(0)|\pi^c(p)\rangle = \frac{1}{2}p_\rho p_\sigma F_0\delta^{dc}.
\end{aligned}
$$

Using these expressions, we deduce from (A.5) the following short-distance behavior of the vertex functions

$$
\begin{aligned}
&\lim_{\lambda\to\infty} (\Gamma_{VA})_{\mu\nu}^{abc}(\lambda q, p) \\
&= \frac{\mathrm{i}}{\lambda q^2}F_0 f^{abc}\Big\{(p\cdot q)\eta_{\mu\nu} - q_\mu p_\nu - q_\nu p_\mu + \frac{1}{\lambda q^2}\Big[q^2 p_\mu p_\nu \\
&+(p\cdot q)^2\eta_{\mu\nu} - (p\cdot q)(q_\mu p_\nu + q_\nu p_\mu)\Big]\Big\} + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \\
&\lim_{\lambda\to\infty} (\Gamma_{VP})_{\mu}^{abc}(\lambda q, p) \\
&= \frac{1}{\lambda q^2}\frac{\langle\bar{\psi}\psi\rangle_0}{F_0} f^{abc}\left\{q_\mu + \frac{2}{3}\frac{(p\cdot q)q_\mu - q^2 p_\mu}{\lambda q^2}\right\} \\
&+\mathcal{O}\left(\frac{1}{\lambda^3}\right), \\
&\lim_{\lambda\to\infty} (\Gamma_{VV})_{\mu\nu}^{abc}(\lambda q, p) \\
&= -\frac{\mathrm{i}}{\lambda q^2}F_0 d^{abc}\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta\left\{1 + \frac{p\cdot q}{\lambda q^2}\right\} + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \\
&\lim_{\lambda\to\infty} (\Gamma_{AA})_{\mu\nu}^{abc}(\lambda q, p) \\
&= -\frac{\mathrm{i}}{\lambda q^2}F_0 d^{abc}\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta\left\{1 + \frac{p\cdot q}{\lambda q^2}\right\} + \mathcal{O}\left(\frac{1}{\lambda^3}\right). \qquad (\mathrm{A}.6)
\end{aligned}
$$

The result for Γ_{VP} contradicts the one given in (55) in [33] where no term of order $1/\lambda^2$ appears [67]. Note that this term is transverse, in accordance with the chiral Ward identities from (2.2). On the other hand, our result for the short-distance expansion of Γ_{VA} agrees with the one given in (58) in [33].

B The resonance Lagrangian

The resonance Lagrangian from [38], which generalizes the one already given in [20], reads (omitting terms including a scalar resonance)

$$\begin{aligned}
\mathcal{L}_{\rm res} &= \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_{VV}^{(2)} + \mathcal{L}_{AA}^{(2)} + \mathcal{L}_{VA}^{(2)},\\
\mathcal{L}_V &= -\frac{1}{4}\left\langle \hat{V}_{\mu\nu}\hat{V}^{\mu\nu} - 2M_V^2\hat{V}_\mu\hat{V}^\mu\right\rangle\\
&\quad -\frac{1}{2\sqrt{2}}\left(f_V\left\langle \hat{V}_{\mu\nu}f_+^{\mu\nu}\right\rangle + \mathrm{i}g_V\left\langle \hat{V}_{\mu\nu}\left[u^\mu,u^\nu\right]\right\rangle\right)\\
&\quad +\mathrm{i}\alpha_V\left\langle \hat{V}_\mu\left[u_\nu,f_-^{\mu\nu}\right]\right\rangle + \beta_V\left\langle \hat{V}_\mu\left[u^\mu,\chi_-\right]\right\rangle\\
&\quad +\mathrm{i}\theta_V\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{V}^\mu u^\nu u^\alpha u^\beta\right\rangle + h_V\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{V}^\mu\left\{u^\nu,f_+^{\alpha\beta}\right\}\right\rangle,\\
\mathcal{L}_A &= -\frac{1}{4}\left\langle \hat{A}_{\mu\nu}\hat{A}^{\mu\nu} - 2M_A^2\hat{A}_\mu\hat{A}^\mu\right\rangle\\
&\quad -\frac{1}{2\sqrt{2}}f_A\left\langle \hat{A}_{\mu\nu}f_-^{\mu\nu}\right\rangle + \mathrm{i}\alpha_A\left\langle \hat{A}_\mu\left[u_\nu,f_+^{\mu\nu}\right]\right\rangle\\
&\quad +\gamma_A^{(1)}\left\langle \hat{A}_\mu u_\nu u^\mu u^\nu\right\rangle + \gamma_A^{(2)}\left\langle \hat{A}_\mu\left\{u^\mu,u^\nu u_\nu\right\}\right\rangle\\
&\quad +\gamma_A^{(3)}\left\langle \hat{A}_\mu u_\nu\right\rangle\langle u^\mu u^\nu\rangle + \gamma_A^{(4)}\left\langle \hat{A}_\mu u^\mu\right\rangle\langle u^\nu u_\nu\rangle\\
&\quad +h_A\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{A}^\mu\left\{u^\nu,f_-^{\alpha\beta}\right\}\right\rangle,\\
\mathcal{L}_{VV}^{(2)} &= \frac{1}{2}\delta_V^{(1)}\left\langle \hat{V}_\mu\hat{V}^\mu u_\nu u^\nu\right\rangle + \frac{1}{2}\delta_V^{(2)}\left\langle \hat{V}_\mu u_\nu\hat{V}^\mu u^\nu\right\rangle\\
&\quad +\frac{1}{2}\delta_V^{(3)}\left\langle \hat{V}_\mu\hat{V}_\nu u^\mu u^\nu\right\rangle + \frac{1}{2}\delta_V^{(4)}\left\langle \hat{V}_\mu\hat{V}_\nu u^\nu u^\mu\right\rangle\\
&\quad +\frac{1}{2}\delta_V^{(5)}\left\langle \hat{V}_\mu u^\mu\hat{V}_\nu u^\nu + \hat{V}_\mu u_\nu\hat{V}^\nu u^\mu\right\rangle + \frac{1}{2}\kappa_V\left\langle \hat{V}_\mu\hat{V}^\mu\chi_+\right\rangle\\
&\quad +\frac{1}{2}\mathrm{i}\phi_V\left\langle \hat{V}_\mu\left[\hat{V}_\nu,f_+^{\mu\nu}\right]\right\rangle\\
&\quad +\frac{1}{2}\sigma_V\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{V}^\mu\left\{u^\nu,\hat{V}^{\alpha\beta}\right\}\right\rangle,\\
\mathcal{L}_{AA}^{(2)} &= \frac{1}{2}\delta_A^{(1)}\left\langle \hat{A}_\mu\hat{A}^\mu u_\nu u^\nu\right\rangle + \frac{1}{2}\delta_A^{(2)}\left\langle \hat{A}_\mu u_\nu\hat{A}^\mu u^\nu\right\rangle\\
&\quad +\frac{1}{2}\delta_A^{(3)}\left\langle \hat{A}_\mu\hat{A}_\nu u^\mu u^\nu\right\rangle + \frac{1}{2}\delta_A^{(4)}\left\langle \hat{A}_\mu\hat{A}_\nu u^\nu u^\mu\right\rangle\\
&\quad +\frac{1}{2}\delta_A^{(5)}\left\langle \hat{A}_\mu u^\mu\hat{A}_\nu u^\nu + \hat{A}_\mu u_\nu\hat{A}^\nu u^\mu\right\rangle + \frac{1}{2}\kappa_A\left\langle \hat{A}_\mu\hat{A}^\mu\chi_+\right\rangle\\
&\quad +\frac{1}{2}\mathrm{i}\phi_A\left\langle \hat{A}_\mu\left[\hat{A}_\nu,f_+^{\mu\nu}\right]\right\rangle + \frac{1}{2}\sigma_A\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{A}^\mu\left\{u^\nu,\hat{A}^{\alpha\beta}\right\}\right\rangle,\\
\mathcal{L}_{VA}^{(2)} &= \mathrm{i}A^{(1)}\left\langle \hat{V}_\mu\left[\hat{A}_\nu,f_-^{\mu\nu}\right]\right\rangle + \mathrm{i}A^{(2)}\left\langle \hat{V}_\mu\left[u_\nu,\hat{A}^{\mu\nu}\right]\right\rangle\\
&\quad +\mathrm{i}A^{(3)}\left\langle \hat{A}_\mu\left[u_\nu,\hat{V}^{\mu\nu}\right]\right\rangle + B\left\langle \hat{V}_\mu\left[\hat{A}^\mu,\chi_-\right]\right\rangle\\
&\quad +H\varepsilon_{\mu\nu\alpha\beta}\left\langle \hat{V}^\mu\left\{\hat{A}^\nu,f_+^{\alpha\beta}\right\}\right\rangle\\
&\quad +\mathrm{i}Z^{(1)}\varepsilon_{\mu\nu\alpha\beta}\left\langle u^\mu u^\nu\left\{\hat{A}^\alpha,\hat{V}^\beta\right\}\right\rangle\\
&\quad +\mathrm{i}Z^{(2)}\varepsilon_{\mu\nu\alpha\beta}\left\langle u^\mu\hat{A}^\nu u^\alpha\hat{V}^\beta\right\rangle, \qquad \text{(B.1)}
\end{aligned}$$

where the vector fields describing the vector and axial-vector resonances have been denoted by $\hat{V}_\mu$ and $\hat{A}_\mu$, respectively. In (B.1) we employed the usual notations [18, 20]

$$\begin{aligned}
\hat{R}_\mu &= \frac{1}{\sqrt{2}}\sum_{a=1}^{8}\hat{R}_\mu^a\lambda^a, \quad \hat{R} = \hat{V},\hat{A},\\
\hat{R}_{\mu\nu} &= \nabla_\mu\hat{R}_\nu - \nabla_\nu\hat{R}_\mu,\\
\nabla_\mu\hat{R} &= \partial_\mu\hat{R} + \left[\Gamma_\mu,\hat{R}\right],
\end{aligned}$$

$$\begin{aligned}
\Gamma_\mu &= \frac{1}{2}\left\{u^\dagger\left[\partial_\mu - \mathrm{i}(v_\mu + a_\mu)\right]u\right.\\
&\quad \left. +\ u\left[\partial_\mu - \mathrm{i}(v_\mu - a_\mu)\right]u^\dagger\right\},\\
f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_{\rm R}^{\mu\nu}u,\\
u_\mu &= \mathrm{i}\left\{u^\dagger\left[\partial_\mu - \mathrm{i}(v_\mu + a_\mu)\right]u\right.\\
&\quad \left. -\ u\left[\partial_\mu - \mathrm{i}(v_\mu - a_\mu)\right]u^\dagger\right\} \equiv \mathrm{i}u^\dagger D_\mu U u^\dagger = u_\mu^\dagger,\\
\chi_\pm &= u^\dagger\chi u^\dagger \pm u\chi^\dagger u,
\end{aligned}$$

where the symbol $\langle M\rangle$ stands for the trace of the 3×3 matrix M. The field u is the square root of the Goldstone boson field, $U = u^2$, whereas v_μ, a_μ and χ denote the external sources.

Note that in the case of bilinear interactions of $\hat{V}_\mu$ and $\hat{A}_\mu$, only terms with at most one trace in flavor space are included in $\mathcal{L}_{\rm res}$, which is compatible with the large-N_C suppression of Zweig rule violating contributions. Furthermore, no pseudoscalar resonances have been incorporated in the resonance Lagrangian in [38]. As stressed in [20], terms from $\mathcal{L}_2 + \mathcal{L}_4$ involving Goldstone bosons have to be added to the resonance Lagrangian as well in order to correctly reproduce the QCD short-distance behavior of certain Green's functions.

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